Multibody Mass Matrix Sensitivity Analysis Using Spatial Operators

Abhinandan Jain & Guillermo Rodriguez

Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, Pasadena, CA 91109

Abstract. This paper discusses an approach for sensitivity analysis of multibody dynamics using spatial operators. The spatial operators are rooted in the function space approach to estimation theory developed in the decades that followed the introduction of the Kalman filter and used extensively by authors to develop a range of results in multibody dynamics. The operators provide a mathematical framework for studying a wide range of analytical and computational problems associated with multi-body system dynamics. This paper focuses on the computation of the sensitivity of the system mass matrix for tree-topology multibody systems and develops an analytical expression for the same using spatial operators. As an application example, the mass matrix sensitivity is used to derive analytical expressions based on composite body inertias for the Christoffel symbols associated with the equations of motion.

1 Introduction

Kalman introduced the notion of a state space, and a recursive filter [Kalman 60] that computes the best estimate of the state from possibly noisy past measurements. The optimal Bryson [Bryson 63] smoother computes the best state estimate using both past and future data. Although several authors seemed to have arrived at similar results at approximately the same time, Kailath [Kailath 70, Kailath 74] was most likely the first to recognize many new techniques. He introduced the "innovations" approach, which when specialized to state space systems was a more advanced way to derive optimal linear estimators such as the Kalman filter. He also recognized the value to estimation theory of powerful mathematical techniques (Gohberg and Krein) to factor positive operators into a product of two closely related integral operators with triangular kernels. The function space approach reached maturity in the work of Balakrishnan [Balakrishnan 77], who introduced the elegant methods of Hilbert space. At the end of this period, we knew how to easily solve very complicated linear filtering problems using linear integral operators, operator factorization methods, and triangular (Volterra) factors.

In the mid 1980's, the authors recognized [Rodriguez 87, Rodriguez 92b, Rodriguez 90] that the equations of mechanical systems had an almost perfect analogy to those of state space linear systems. Discovery of this analogy allowed the use in mechanics of very advanced methods and computational architectures (Kalman, Bryson, Riccati, etc.) that had emerged from estimation theory. Also, the parallels led to the introduction of *spatial operators* to succinctly describe at a highlevel complex multibody dynamics quantities and relationships. The rich structural properties of the spatial operators and the ability to do mathematics with them to derive new reletionships and computational algorithms led to the coining of the **spatial operator algebra** term for this multibody dynamics framework. An overview of the spatial operator algebra can be found in [Rodriguez 91, Jain 91, Jain 00]. Some of the contributions of the spatial operator approach include the closed-form expressions for the mass matrix, the $O(\mathcal{N})$ algorithms for the computation of the *operational space inertia matrix* [Kreutz-Delgado 92], the dynamics of under-actuated systems [Jain 93a], diagonalized dynamics formulations [Jain 95] etc.

In this paper we describe our recent results which use an analytical approach for tree-topology multibody dynamics sensitivity computations using spatial operators. Sensitivity computations arise in problems involving optimization, linearization, nonlinear analysis and control of multibody systems. Example multibody applications where such sensitivity computations are useful can be found in [Jain 93b, Jain 95]. The key role of the system mass matrix in multibody dynamics implies that its sensitivity plays a central role in most multibody sensitivity analysis. The mass matrix sensitivities also underly the velocity dependent gyroscopic and Coriolis terms that appear in the Lagrangian form of the equations of motion.

In practice, due to the complexity of the dynamics quantities, numerical differentiation techniques are often utilized for such multibody sensitivity computations. Not only are these techniques non-robust, they also introduce errors and are computationally expensive. In this paper we focus less on the computational issues, and more on using the spatial operator approach for sensitivity computations to develop new relationships. In particular we establish connections between the mass matrix sensitivities and the composite rigid body inertias which play a key role in inverse dynamics problems. As illustration, these relationships are used to develop closed-form expressions for the well know Christoffel symbols that are related to the non-linear velocity dependent gyroscopic and Coriolis terms in the equations of motion. One interpretation of this analysis is the natural bridging between the abstract Langrangian expressions for these velocity dependent terms and the component level expressions that can be used for computations. References [Jain 99, Bestle 92a, Bestle 92b, Chang 85, Eberhard 96, Hsu 01, Haug 84, Serban 98] contain additional detailed discussion of the computational and other aspects of multibody sensitivities.

The promise of the spatial operator approach is:

- It is applicable to large-dimensional systems, is accurate and is computationally efficient, as it makes use of the highly developed Kalman filter computational architecture. This adds an enormous amount of algorithmic and computational robustness to the evaluation of analytical terms, the sensitivity of the mass matrix with respect to any given joint angle for example, that are otherwise typically evaluated by symbolic or numerical differentiation.

- The sensitivities of the spatial operators are expressed in terms of the spatial operators themselves, which implies that the sensitivities for the mass matrix can be evaluated by spatial recursions quite similar to those associated with recursive evaluation of the mass matrix itself. This property also implies that if necessary higher-order sensitivities could be computed by applying the spatial operator sensitivity formulas derived here repeatedly.

Such exact methods for computing mass matrix system model sensitivities, in contrast to the use of approximate methods, are relatively easy to implement with spatial recursions that are already being used to implement the evaluation of the mass matrix itself. There is no need to introduce additional numerical computations that may be specified within a framework that does not exactly match that which is being used to evaluate the mass matrix. Instead, the spatial recursions that are used to evaluate the mass matrix itself, or alternatively to implement the original system equations of motion, provide the computational structures to evaluate the system mass matrix sensitivities. This feature, the commonality in the spatial recursions needed for both the mass matrix and its sensitivities, also applies for efficient evaluation of the velocity dependent Coriolis term in the original equations of motion. An added benefit of using spatial operators to specify the spatial recursions is that the system mass matrix sensitivities can be computed with exactly the same spatial operators. This is a unique feature of the spatial-operator-based method that is being described here.

The set of spatially recursive methods embodied by the spatial operator algebra have been applied to a large variety of systems including: analytical design, software development, real-time hardware in the loop simulation, and flight operations for several planetary spacecraft [Jain 92a, Biesiadecki 96]: dual-arm robotic space systems; and large-scale simulation of molecular systems for use in addressing such problems as the analysis of protein-folding and new medicinal drug design [Jain 93c]. While this present paper focuses primarily on theoretical issues, the motivation for the work is drawn from many practical applications [Jain 92a, Biesiadecki 96, Jain 93c] where the methods are currently in use.

1.1 Overview of Spatial Operators for Serial Chain Systems

The aim of this subsection is to summarize briefly the essential ideas underlying spatial operators leading up to the Newton-Euler Operator Factorization $\mathcal{M}(\theta) = \boldsymbol{H}\phi\boldsymbol{M}\phi^*\boldsymbol{H}^*$ of the manipulator mass matrix. While this is done here for a serial chain manipulator, the factorization results apply to a much more general class of complex joint-connected mechanical systems, including tree configurations with flexible links and joints [Jain 92b].

Consider a serial manipulator with $\mathcal{N}$ rigid links in Figure 1. . The links are numbered in increasing order from tip to base. The outer-most link is link 1, and the inner-most link is link $\mathcal{N}$. The overall number of degrees-of-freedom for the manipulator is $\mathcal{N}$. There are two joints attached to the k^{th} link. A coordinate frame $\mathcal{O}_k$ is attached to the inboard joint, and another frame $\mathcal{O}^+_{k-1}$ is attached to the outboard joint. Frame $\mathcal{O}_k$ is also the body frame for the k^{th} link. The k^{th} joint connects the $(k+1)^{st}$ and k^{th} links, and its motion is defined as the motion of frame $\mathcal{O}_k$ with respect to frame $\mathcal{O}^+_k$. When applicable, the free-space motion of a manipulator is modeled by attaching a 6 degree-of-freedom joint between the base

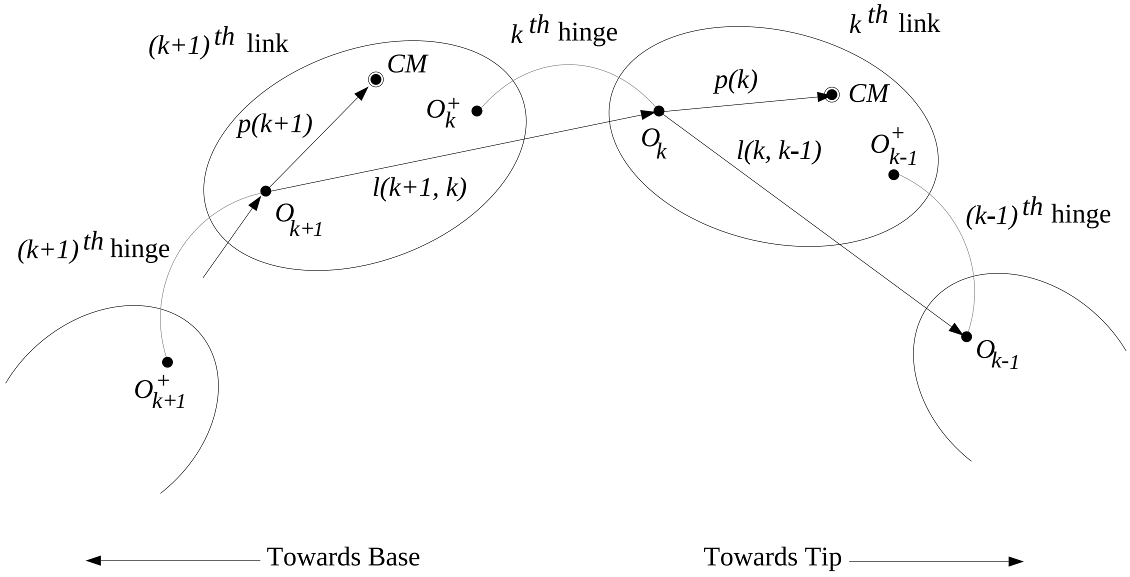

Figure 1: Illustration of links and joints in a serial rigid body system

link and the inertial frame about which the free-space motion occurs. However, in this paper, without loss of generality and for the sake of notational simplicity, all joints are assumed to be single rotational degree-of-freedom joints with the k^{th} joint coordinate given by $\theta(k)$. Extension to joints with more rotational and translational degrees-of-freedom is easy [Rodriguez 92a].

The transformation operator $\phi(k, k-1)$ between the $\mathcal{O}_{k-1}$ and $\mathcal{O}_k$ frames is

$$\phi(k, k-1) \quad = \quad \begin{pmatrix} \boldsymbol{I}_3 & \tilde{l}(k, k-1) \\ 0 & \boldsymbol{I}_3 \end{pmatrix} \in \mathcal{R}^{6\times 6}$$

where $l(k, k-1)$ is the vector from frame $\mathcal{O}_k$ to frame $\mathcal{O}_{(k-1)}$, and $\tilde{l}(k, k-1) \in \mathcal{R}^{3\times 3}$ is the skew–symmetric matrix associated with the cross-product operation.

The spatial velocity of the k^{th} body frame $\mathcal{O}_k$ is $V(k) = [\omega^*(k), v^*(k)]^* \in \mathcal{R}^6$, where $\omega(k)$ and $v(k)$ are the angular and linear velocities of $\mathcal{O}_k$. With $h(k) \in \mathcal{R}^3$ denoting the k^{th} joint axis vector, $\boldsymbol{H}(k) = [h^*(k), 0] \in \mathcal{R}^1 \times \mathcal{R}^6$ denotes the joint map matrix for the joint, and the relative spatial velocity across the k^{th} joint is $\boldsymbol{H}^*(k)\dot{\theta}(k)$. The spatial force of interaction $f(k)$ across the k^{th} joint is $f(k)$= $[N^*(k), F^*(k)]^* \in \mathcal{R}^6$, where $N(k)$ and $F(k)$ are the moment and force components respectively. The 6×6 spatial inertia matrix $\boldsymbol{M}(k)$ of the k^{th} link in the coordinate frame $\mathcal{O}_k$ is

$$\boldsymbol{M}(k) \quad = \quad \begin{pmatrix} \mathcal{J}(k) & m(k)\tilde{p}(k) \\ -m(k)\tilde{p}(k) & m(k)\boldsymbol{I}_3 \end{pmatrix}$$

where $m(k)$ is the mass, $p(k)\in \mathcal{R}^3$ is the vector from $\mathcal{O}_k$ to the k^{th} link center of mass, and $\mathcal{J}(k)\in\mathcal{R}^{3\times 3}$ is the rotational inertia of the k^{th} link about $\mathcal{O}_k$. $\boldsymbol{I}_3$ is the 3×3 unit matrix.

The recursive Newton–Euler equations are [Luh 80, Rodriguez 87]

$$\begin{cases} \quad V(\mathcal{N}+1)=0; \qquad \alpha(\mathcal{N}+1)=0 \\ \textbf{for } \boldsymbol{k} = \boldsymbol{\mathcal{N}} \cdots \mathbf{1} \\ \quad V(k) = \phi^*(k+1,k)V(k+1) + \boldsymbol{H}^*(k)\dot{\theta}(k) \\ \quad \alpha(k) = \phi^*(k+1,k)\alpha(k+1) + \boldsymbol{H}^*(k)\ddot{\theta}(k) + a(k) \\ \textbf{end loop} \end{cases}$$

$$\begin{cases} \quad f(0)=0 \\ \textbf{for } \boldsymbol{k} = \mathbf{1} \cdots \boldsymbol{\mathcal{N}} \\ \quad f(k) = \phi(k,k-1)f(k-1) + \boldsymbol{M}(k)\alpha(k) + b(k) \\ \quad \boldsymbol{T}(k) = \boldsymbol{H}(k)f(k) \\ \textbf{end loop} \end{cases}$$

where $\boldsymbol{T}(k)$ is the applied moment at joint k. The nonlinear, velocity dependent terms $a(k)$ and $b(k)$ are respectively the Coriolis acceleration and the gyroscopic force terms for the k^{th} link.

The "stacked" notation $\theta = \mathrm{col}\left\{\theta(k)\right\} \in \mathcal{R}^{\mathcal{N}}$ is used to simplify the above recursive Newton-Euler equations. This notation [Rodriguez 91] eliminates the arguments k associated with the individual links by defining composite vectors, such as θ, which apply to the entire manipulator system. We define

$$\begin{array}{ll} \boldsymbol{T} = \mathrm{col}\left\{\boldsymbol{T}(k)\right\} \in \mathcal{R}^{\mathcal{N}} & V = \mathrm{col}\left\{V(k)\right\} \in \mathcal{R}^{6\mathcal{N}} \\ f = \mathrm{col}\left\{f(k)\right\} \in \mathcal{R}^{6\mathcal{N}} & \alpha = \mathrm{col}\left\{\alpha(k)\right\} \in \mathcal{R}^{6\mathcal{N}} \\ a = \mathrm{col}\left\{a(k)\right\} \in \mathcal{R}^{6\mathcal{N}} & b = \mathrm{col}\left\{b(k)\right\} \in \mathcal{R}^{6\mathcal{N}} \end{array}$$

In this notation, the equations of motion are [Rodriguez 87, Rodriguez 92b]:

$$V = \phi^*\boldsymbol{H}^*\dot{\theta}; \qquad \alpha = \phi^*[\boldsymbol{H}^*\ddot{\theta} + a] \tag{1.1}$$

$$f = \phi[\boldsymbol{M}\alpha + b]; \qquad \boldsymbol{T} = \boldsymbol{H}f = \mathcal{M}\ddot{\theta} + \mathcal{C} \tag{1.2}$$

where the mass matrix $\mathcal{M}(\theta) = \boldsymbol{H}\phi\boldsymbol{M}\phi\boldsymbol{H}^*$; $\mathcal{C}(\theta,\dot{\theta}) = \boldsymbol{H}\phi[\boldsymbol{M}\phi^* a + b] \in \mathcal{R}^{\mathcal{N}}$ is the Coriolis term; $\boldsymbol{H} = \mathrm{diag}\left\{\boldsymbol{H}(k)\right\} \in \mathcal{R}^{\mathcal{N}\times 6\mathcal{N}}$; $\boldsymbol{M} = \mathrm{diag}\left\{\boldsymbol{M}(k)\right\} \in \mathcal{R}^{6\mathcal{N}\times 6\mathcal{N}}$; and $\phi \in \mathcal{R}^{6\mathcal{N}\times 6\mathcal{N}}$

$$\phi = (\boldsymbol{I} - \mathcal{E}_\phi)^{-1} = \begin{pmatrix} \boldsymbol{I} & 0 & \dots & 0 \\ \phi(2,1) & \boldsymbol{I} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \phi(n,1) & \phi(n,2) & \dots & \boldsymbol{I} \end{pmatrix} \tag{1.3}$$

with $\phi(i,j) = \phi(i,i-1)\cdots\phi(j+1,j)$ for $i > j$. The shift operator $\mathcal{E}_\phi \in \mathcal{R}^{6\mathcal{N}\times 6\mathcal{N}}$ is defined as

$$\mathcal{E}_\phi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \phi(2,1) & 0 & \dots & 0 & 0 \\ 0 & \phi(3,2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \phi(\mathcal{N},\mathcal{N}-1) & 0 \end{pmatrix} \tag{1.4}$$

Using spatial operators one can obtain operator factorizations of the mass matrix and its inverse as follows:

Identity 1.1

$$\begin{aligned}\mathcal{M} &= \boldsymbol{H}\phi\boldsymbol{M}\phi^*\boldsymbol{H}^* \\ &= [\boldsymbol{I}+\boldsymbol{H}\phi\boldsymbol{K}]\boldsymbol{D}[\boldsymbol{I}+\boldsymbol{H}\phi\boldsymbol{K}]^* \\ [\boldsymbol{I}+\boldsymbol{H}\phi\boldsymbol{K}]^{-1} &= \boldsymbol{I}-\boldsymbol{H}\psi\boldsymbol{K} \\ \mathcal{M}^{-1} &= [\boldsymbol{I}-\boldsymbol{H}\psi\boldsymbol{K}]^*\boldsymbol{D}^{-1}[\boldsymbol{I}-\boldsymbol{H}\psi\boldsymbol{K}]\end{aligned}$$

These identifies have been used extensively [Rodriguez 87, Rodriguez 92b, Rodriguez 90, Rodriguez 91, Jain 91, Jain 00, Jain 93b, Jain 95], to develop a variety of spatially recursive algorithms for forward dynamics, for both rigid and flexible multi-body systems of arbitrarily specified topologies, as well as closed-form analytical expressions for the inverse of the mass matrix. The spatial operators ψ, $\boldsymbol{D}$ correspond to a suitably defined spatially recursive Kalman filter, with the spatial operator $\boldsymbol{K}$ representing the Kalman gain for this filter. We also refer to these operators ψ, $\boldsymbol{D}$ and $\boldsymbol{K}$ as"articulated" quantities, because of their relationship to the articulated inertias first introduced by [Featherstone 83].

The approach presented here to compute sensitivities for the mass matrix and the spatial operators embedded in it has been extended to be able to compute sensitivities of the articulated operators. We have used [Jain 95] such sensitivities of articulated operators to evaluate explicitly the Coriolis term in a diagonalized version of Lagrange's equations of motion. However, it is beyond the scope of this article to address the issue of how to compute sensitivities for these articulated operators. We will therefore not go any further in this direction, but instead will focus on the process of computing the sensitivity of spatial operators associated with the mass matrix itself. For the purpose of this discussion, we focus attention on serial chains with single-degree-of-freedom hinges. We will maintain this focus in the rest of the paper, although the generalization to general tree-topology systems and hinges in straightforward.

2 Preliminary Notation

We introduce some preliminary notation that will simplify the evaluation of the spatial operators embedded in the mass matrix. Specifically, we introduce: 1) a shift-operator $\mathbb{S}$ and develop some of its properties; 2) a 6-dimensional skew-symmetric operator, analogous to the cross-product of ordinary 3-dimensional vector algebra; and 3) a set of "pick-off" operators $\mathbb{H}^i$, $\mathbb{H}^i_s$, and $\mathbb{H}^i_\delta$ that, when applied to any given spatial vector, have the effect of "picking-off" or operating only on quantities associated with the i^{th} hinge in this vector. While introducing these quantities may appear to be somewhat arbitrary at this stage, they will subsequently prove themselves quite useful in streamlining the sensitivity computations that are the focus of this paper.

2.1 The Shift operator $\mathbb{S}$

We first introduce and define the *shift operator*, $\mathbb{S} \in \mathcal{R}^{6\mathcal{N}\times 6\mathcal{N}}$ consisting of $\mathcal{R}^{6\times 6}$ block elements with the only non-zero ones being the identity $\mathcal{R}^{6\times 6}$ elements along the first sub-diagonal. Some useful properties of the shift operator $\mathbb{S}$ are defined in the following lemma.

Lemma 2.1 **: Properties of the shift operator $\mathbb{S}$**

Given block diagonal matrices A and B, the following relationships hold:

$$(\mathbb{S}A\mathbb{S}^*)\mathbb{S}B = \mathbb{S}AB$$

$$(\mathbb{S}^*A\mathbb{S})\mathbb{S}^*B = \mathbb{S}^*AB$$

$$A\mathbb{S}^*(\mathbb{S}B\mathbb{S}^*) = AB\mathbb{S}^*$$

$$A\mathbb{S}(\mathbb{S}^*B\mathbb{S}) = AB\mathbb{S}$$

$$(\mathbb{S}A\mathbb{S}^*)(\mathbb{S}B\mathbb{S}^*) = \mathbb{S}AB\mathbb{S}^*$$

$$(\mathbb{S}^*A\mathbb{S})(\mathbb{S}^*B\mathbb{S}) = \mathbb{S}^*AB\mathbb{S}$$

Special Cases:

$$(\mathbb{S}\mathbb{S}^*)\mathbb{S}A = \mathbb{S}A \qquad \mathbb{S}A(\mathbb{S}^*\mathbb{S}) = \mathbb{S}A$$

$$(\mathbb{S}^*\mathbb{S})\mathbb{S}^*A = \mathbb{S}^*A \qquad \mathbb{S}^*A(\mathbb{S}\mathbb{S}^*) = \mathbb{S}^*A$$

$$(\mathbb{S}^*\mathbb{S})A\mathbb{S}^* = A\mathbb{S}^* \qquad A(\mathbb{S}^*\mathbb{S})\mathbb{S}^* = A\mathbb{S}^*$$

$$(\mathbb{S}\mathbb{S}^*)A\mathbb{S} = A\mathbb{S} \qquad A(\mathbb{S}\mathbb{S}^*)\mathbb{S} = A\mathbb{S}$$

Proof: *Use direct evaluation to verify these identities.* ∎

2.2 The spatial vector cross product

With $z \triangleq \begin{bmatrix} x \\ y \end{bmatrix}$ and $c \triangleq \begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathcal{R}^6$, define the **6-dimensional cross-product** operation as:

$$z \times c = \tilde{z}c = \begin{pmatrix} \tilde{x}a \\ \tilde{y}a + \tilde{x}b \end{pmatrix} \quad \text{where} \quad \tilde{z} \triangleq \begin{pmatrix} \tilde{x} & \mathbf{0}_3 \\ \tilde{y} & \tilde{x} \end{pmatrix} \tag{2.5}$$

For 3-vectors, the $\tilde{x}$ terminology denotes the standard 3 by 3 skew-symmetric matrix associated with the 3-vector cross-product operation.

The following lemma provides an intuitive rationale for extending the “cross-product” terminology above from 3-vectors to 6-vectors.

Lemma 2.2 **: Spatial vector cross-product identities**

We have the following identities for the spatial vector cross-product:

$$\tilde{A}A = 0 \qquad \textit{and} \qquad \tilde{A}B = -\tilde{B}A \tag{2.6}$$

where A and B are two given spatial vectors.

Proof: *The identity $\tilde{A}A$ follows from the skew-symmetry of $\tilde{A}$. The remaining are established by verification for arbitrary vectors A and B.* ∎

Note that the above *skew-symmetric* property holds even though unlike in the 3-dimensional case, the $\tilde{z}$ matrix for 6-vectors is *not* skew-symmetric! The differential geometric interpretations and properties of $\phi^*(\cdot)$, and of the 6-dimensional cross-product, are discussed further in [Li 89, Murray 94]. The "˜" operator defined above is a natural generalization to spatial quantities of the 3-dimensional cross-product operator.

For notational convenience we also define the operation $\mathcal{S}[z]$ which is closely related to the cross-product operation as follows:

$$\mathcal{S}\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] \triangleq \begin{pmatrix} \tilde{x} & \tilde{y} \\ 0 & \tilde{x} \end{pmatrix} = -\left\{\widetilde{\begin{pmatrix} x \\ y \end{pmatrix}}\right\}^* \tag{2.7}$$

where x and y are arbitrary 3-vectors.

Also, with the **skew-symmetric matrix** defined as

$$\mathcal{Q}[z] \triangleq \begin{pmatrix} \tilde{x} & \tilde{y} \\ \tilde{y} & \mathbf{0}_3 \end{pmatrix} \tag{2.8}$$

the following relationships can be verified:

$$\tilde{A}^*B = \mathcal{Q}[B]A = -\mathcal{Q}^*[B]A \quad \text{or} \quad \mathcal{S}[A]\,B = -\mathcal{Q}[B]A \tag{2.9}$$

Thus the $\mathcal{Q}$ operator is in a sense the adjoint of the ˜ operator for spatial vectors. Just as the 3-dimensional cross-product defines the Lie bracket for the $so(3)$ Lie-algebra for the $SO(3)$ Lie group, the 6-dimensional cross-product defines the Lie bracket for the $\boldsymbol{ad}$ Lie algebra associated with the $\boldsymbol{Ad}$ Lie group made up of the $\phi^*(\cdot)$ elements. The identities in Eq. 2.6 are just special cases of identities that involve the Lie bracket operation.

Lemma 2.3 **: Identities involving $\mathcal{Q}(\cdot)$ and $\phi(\cdot,\cdot)$**

For an arbitrary 6-vector X, we have the following identities:

$$\begin{aligned} \mathcal{Q}\left[\phi(l)X\right] &= \phi(l)\mathcal{Q}[X]\phi^*(l) \\ \mathcal{S}\left[\phi^*(l)X\right] &= \phi^{-1}(l)\mathcal{S}\left[X\right]\phi(l) \\ [\widetilde{\phi^*(l)X}] &= \phi^*(l)\tilde{X}\phi^{-*}(l) \\ [\widetilde{\phi^*(l)X}]\phi^*(l) &= \phi^*(l)\tilde{X} \end{aligned} \tag{2.10}$$

The second identity parallels the well known identity $\widetilde{\mathfrak{R}l} = \mathfrak{R}\tilde{l}\mathfrak{R}^*$, where $\mathfrak{R}$ is a rotational matrix and l is a 3-vector. This is a well know differential geometric identity that applies in general to $\boldsymbol{Ad}$ and $\boldsymbol{ad}$ representations.

Proof: *The first two identities can be established by verification for arbitrary 3-vectors l and X. The last one is a simple restatement of the second identity.* ∎

2.3 The $\mathbb{H}^i$, $\mathbb{H}^i_s$, and $\mathbb{H}^i_\delta$ operators

We define $\mathbb{H}(i)$ as

$$\mathbb{H}(i) \ \stackrel{\triangle}{=} \ \mathcal{S}^* \left[\boldsymbol{H}^*(i)\right] = \begin{pmatrix} \tilde{h}(i) & 0 \\ 0 & \tilde{h}(i) \end{pmatrix} \tag{2.11}$$

$\mathbb{H}^i_s$ is the block diagonal matrix defined as $\mathbb{H}^i_s(k,k) = \mathbb{H}(i)\delta_{k<i}$, i.e.

$$\mathbb{H}^i_s(k,k) = \begin{cases} \mathbb{H}(i) & \text{for } k > i \\ 0 & \text{for } k \leq i \end{cases} \tag{2.12}$$

We similarly also define the block diagonal matrices $\mathbb{H}^i$ and $\mathbb{H}^i_\delta$ as having $\mathbb{H}(i)$ along the block diagonal in the following manner:

$$\mathbb{H}^i(k,k) = \mathbb{H}(i)\delta_{k\leq i}, \quad \text{and} \quad \mathbb{H}^i_\delta(k,k) = \mathbb{H}(i)\delta_{k=i} \tag{2.13}$$

In the above, the δ_{cond} notation is defined such that its value is 1 if $cond$ is true and is 0 otherwise. There is an important new quantity in this result, and it has a simple physical interpretation. The matrix $\mathbb{H}^i_\delta$ is the $6\mathcal{N} \times 6\mathcal{N}$ matrix whose elements are all zero, except for a single 6×6 block $\mathbb{H}(i)$ at the i^{th} location on the diagonal. The index i corresponds to the joint-angle θ_i with respect to which the sensitivity $\mathcal{M}_{\theta_i}$ is being taken.

Note that

$$\mathbb{H}^i = \mathbb{H}^i_s + \mathbb{H}^i_\delta, \quad \text{and} \quad \mathbb{H}^i_s = \mathbb{S}^*\mathbb{H}^i\mathbb{S} \tag{2.14}$$

Also, $\mathbb{H}^i$, $\mathbb{H}^i_s$ and $\mathbb{H}^i_\delta$ are all skew-symmetric.

Lemma 2.4 **: Composition of $\mathbb{H}^i_\delta$ etc. with arbitrary matrices.**

For a given matrix X we have that,

$$\begin{aligned}
[\mathbb{H}^i_s X](k,j) &= \mathbb{H}(i)X(k,j)\delta_{k<i} \\
[X\mathbb{H}^i_s](k,j) &= X(k,j)\mathbb{H}(i)\delta_{j<i} \\
[\mathbb{H}^i X](k,j) &= \mathbb{H}(i)X(k,j)\delta_{k\leq i} \\
[X\mathbb{H}^i](k,j) &= X(k,j)\mathbb{H}(i)\delta_{j\leq i} \\
[\mathbb{H}^i_\delta X](k,j) &= \mathbb{H}(i)X(k,j)\delta_{k=i} \\
[X\mathbb{H}^i_\delta](k,j) &= X(k,j)\mathbb{H}(i)\delta_{j=i} \\
[X\mathbb{H}^i_\delta Y](k,j) &= X(k,i)\mathbb{H}(i)Y(i,j)
\end{aligned} \tag{2.15}$$

Proof: *These identities are established by simply evaluating the products on the right hand side of the equations.* ∎

Define

$$\tilde{\Omega}(k) \triangleq \begin{pmatrix} \tilde{\omega}(k) & 0 \\ 0 & \tilde{\omega}(k) \end{pmatrix} \tag{2.16}$$

and

$$\tilde{\Omega} = \sum_{i=1}^{n} \mathbb{H}^i \dot{\theta}(i), \quad \tilde{\Omega}_s = \sum_{i=1}^{n} \mathbb{H}_s^i \dot{\theta}(i), \quad \tilde{\Omega}_\delta = \sum_{i=1}^{n} \mathbb{H}_\delta^i \dot{\theta}(i)$$

$\tilde{\Omega}$ is the spatial cross product matrix associated with the spatial vector $\Omega(k)$, where $\Omega(k)$ is defined as:

$$\Omega(k) \triangleq \begin{bmatrix} \omega(k) \\ 0 \end{bmatrix} \tag{2.17}$$

Note that

$$\tilde{\Omega} = \tilde{\Omega}_s + \tilde{\Omega}_\delta, \quad \text{and} \quad \tilde{\Omega}_s = \mathbb{S}^* \tilde{\Omega} \mathbb{S} \tag{2.18}$$

3 Sensitivity Computations

Given the generalized coordinates vector θ and a multi-valued function $g(\theta)$, our general approach to computing its sensitivity will be to first compute an expression for its time derivative $\dot{g}(\theta)$ and then use the relationship

$$\dot{g}(\theta) = \frac{\partial g(\theta)}{\partial \theta} \dot{\theta}$$

to obtain $\frac{\partial g(\theta)}{\partial \theta_i}$ from the i^{th} column of $\frac{\partial g(\theta)}{\partial \theta}$.

3.1 Sensitivities of $\phi(k+1,k)$, $\boldsymbol{H}(k)$ and $\boldsymbol{M}(k)$

Having defined a set of useful quantities that will play a key role in streamlining subsequent derivations, we now begin a process of systematically evaluating spatial operator sensitivities at two distinct layers of abstraction. First, we derive sensitivities for spatial operators defined at each link, $\phi(k+1,k)$ for example, and then we derive similar sensitivities for spatial operators, ϕ for example, defined over the entire span of the serial-chain system.

Lemma 3.1 **: Time derivatives of $\phi(k+1,k)$, $\boldsymbol{H}(k)$ and $\boldsymbol{M}(k)$**

We have that

$$\dot{\boldsymbol{M}}(k) = \tilde{\Omega}(k)\boldsymbol{M}(k) - \boldsymbol{M}(k)\tilde{\Omega}(k) \tag{3.19}$$

$$\dot{\boldsymbol{H}}^*(k) = \tilde{\Omega}(k+1)\boldsymbol{H}^*(k) \tag{3.20}$$

$$\dot{\phi}(k+1,k) = \tilde{\Omega}(k+1)\phi(k+1,k) - \phi(k+1,k)\tilde{\Omega}(k+1) \tag{3.21}$$

Proof:

$$\dot{\phi}(k+1,k) = \begin{pmatrix} 0 & \tilde{\omega}(k+1)\widetilde{\ell(k+1,k)} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{\omega}(k+1)\tilde{\ell}(k+1,k) - \tilde{\ell}(k+1,k)\tilde{\omega}(k+1) \\ 0 & 0 \end{pmatrix} \tag{3.22}$$

Also,

$$\dot{\boldsymbol{H}}^*(k) = \begin{bmatrix} \tilde{\omega}(k+1)h(k) \\ 0 \end{bmatrix} = \tilde{\Omega}(k+1)\boldsymbol{H}^*(k) \tag{3.23}$$

Also,

$$\dot{\boldsymbol{M}}(k) = \begin{pmatrix} \tilde{\omega}(k)\mathcal{J}(k) - \mathcal{J}(k)\tilde{\omega}(k) & m(k)[\tilde{\omega}(k)p(k)]^{\sim} \\ -m(k)[\tilde{\omega}(k)p(k)]^{\sim} & 0 \end{pmatrix} = \tilde{\Omega}(k)\boldsymbol{M}(k) - \boldsymbol{M}(k)\tilde{\Omega}(k) \tag{3.24}$$

∎

Lemma 3.2 **: Sensitivities of $\phi(k+1,k)$, $\boldsymbol{H}(k)$ and $\boldsymbol{M}(k)$**

$$\begin{aligned} [\phi(k+1,k)]_{\theta_i} &= [\mathbb{H}(i)\phi(k+1,k) - \phi(k+1,k)\mathbb{H}(i)] \cdot \delta_{k<i} \\ &= \begin{cases} 0 & \textit{for} \quad k \geq i \\ \mathbb{H}(i)\phi(k+1,k) - \phi(k+1,k)\mathbb{H}(i) & \textit{for} \quad k > i \end{cases} \end{aligned} \tag{3.25}$$

$$[\boldsymbol{H}^*(k)]_{\theta_i} = \mathbb{H}(i)\boldsymbol{H}^*(k)\delta_{k<i} = \begin{cases} 0 & \textit{for} \quad k \geq i \\ \begin{bmatrix} \tilde{h}(i)h(k) \\ 0 \end{bmatrix} & \textit{for} \quad k < i \end{cases} \tag{3.26}$$

$$[\boldsymbol{M}(k)]_{\theta_i} = [\mathbb{H}(i)\boldsymbol{M}(k) - \boldsymbol{M}(k)\mathbb{H}(i)]\delta_{k\leq i} = \begin{cases} 0 & \textit{for} \quad k > i \\ \mathbb{H}(i)\boldsymbol{M}(k) - \boldsymbol{M}(k)\mathbb{H}(i) & \textit{for} \quad k \leq i \end{cases} \tag{3.27}$$

Proof: *Follow directly from Lemma 3.1.* ∎

3.2 Operator sensitivities of ϕ, $\boldsymbol{H}$, $\boldsymbol{M}$

Now we proceed to compute sensitivities for the operators $\phi, \boldsymbol{H}, \boldsymbol{M}$ which together constitute the Newton-Euler factorization $\mathcal{M} = \boldsymbol{H}\phi\boldsymbol{M}\phi^*\boldsymbol{H}^*$ of the mass matrix.

Define the operator Δ_ϕ as follows.

$$\Delta_\phi = \begin{pmatrix} \phi(2,1) & 0 & \dots & \\ 0 & \phi(3,2) & & \\ \vdots & & \ddots & \\ 0 & \dots & \dots & \phi(n+1,n) \end{pmatrix} \tag{3.28}$$

Note that

$$\mathcal{E}_\phi = \mathbb{S}\Delta_\phi \tag{3.29}$$

Lemma 3.3 **: Time Derivatives of Spatial Operators**

$$\dot{\Delta}_\phi = \tilde{\Omega}_s \Delta_\phi - \Delta_\phi \tilde{\Omega}_s \tag{3.30}$$

$$\dot{\mathcal{E}}_\phi = \tilde{\Omega} \mathcal{E}_\phi - \mathcal{E}_\phi \tilde{\Omega}_s \tag{3.31}$$

$$\dot{\boldsymbol{H}}^* = \tilde{\Omega}_s \boldsymbol{H}^* \tag{3.32}$$

$$\dot{\boldsymbol{M}} = \tilde{\Omega} \boldsymbol{M} - \boldsymbol{M} \tilde{\Omega} \tag{3.33}$$

$$\dot{\phi} = \phi \tilde{\Omega} \tilde{\phi} - \tilde{\phi} \tilde{\Omega}_s \phi = \phi \tilde{\Omega}_\delta \phi + \tilde{\Omega}_s \phi - \phi \tilde{\Omega} \tag{3.34}$$

Proof: *Eq. 3.30 can be derived by assembling the component time derivatives of Eq. 3.28 from Eq. 3.21. Eq. 3.32 follows by applying the identities in Lemma 2.1 to Eq. 3.30. Eq. 3.32 and Eq. 3.33 are simply matrix versions of Eq. 3.20 and Eq. 3.19 respectively. For Eq. 3.34 we have that*

$$\dot{\phi} = -\phi \dot{\overline{\phi^{-1}}} \phi = -\phi \dot{\overline{[I - \mathcal{E}_\phi]}} \phi = \phi \dot{\mathcal{E}}_\phi \phi = \phi [\tilde{\Omega} \mathcal{E}_\phi - \mathcal{E}_\phi \tilde{\Omega}_s] \phi = \phi \tilde{\Omega} \tilde{\phi} - \tilde{\phi} \tilde{\Omega}_s \phi$$

∎

Lemma 3.4 **: Operator sensitivities of ϕ, $\boldsymbol{H}$, $\boldsymbol{M}$**

$$[\Delta_\phi]_{\theta_i} = \mathbb{H}^i_s \Delta_\phi - \Delta_\phi \mathbb{H}^i_s \tag{3.35}$$

$$[\mathcal{E}_\phi]_{\theta_i} = \mathbb{H}^i \mathcal{E}_\phi - \mathcal{E}_\phi \mathbb{H}^i_s \tag{3.36}$$

$$[\phi]_{\theta_i} = \phi \mathbb{H}^i_\delta \phi - \phi \mathbb{H}^i + \mathbb{H}^i_s \phi \tag{3.37}$$

$$[\phi]_{\theta_i}(k,j) = [\phi(k,i)\mathbb{H}(i)\phi(i,j)\delta_{k\geq i} - \phi(k,j)\mathbb{H}(i) + \mathbb{H}(i)\phi(k,j)\delta_{k<i}]\delta_{j<i} \tag{3.38}$$

$$[\boldsymbol{H}^*]_{\theta_i} = \mathbb{H}^i_s \boldsymbol{H}^* \tag{3.39}$$

$$[\boldsymbol{M}]_{\theta_i} = \mathbb{H}^i \boldsymbol{M} - \boldsymbol{M} \mathbb{H}^i \tag{3.40}$$

Proof:

$$[\mathcal{E}_\phi]_{\theta_i} = \mathbb{S}[\Delta_\phi]_{\theta_i}$$

Since $\phi = [I - \mathcal{E}_\phi]^{-1}$

$$[\phi]_{\theta_i} = -\phi[\phi^{-1}]_{\theta_i}\phi = \phi[\mathbb{H}^i \mathcal{E}_\phi - \mathcal{E}_\phi \mathbb{H}^i_s]\phi = \phi \mathbb{H}^i \tilde{\phi} - \tilde{\phi} \mathbb{H}^i_s \phi = \phi \mathbb{H}^i_\delta \phi - \phi \mathbb{H}^i + \mathbb{H}^i_s \phi$$

$$\begin{aligned}[\phi]_{\theta_i}(k,j) &= \phi(k,i)\mathbb{H}(i)\phi(i,j)\delta_{k\geq i\geq j} - \phi(k,j)\mathbb{H}(i)\delta_{j\leq i} + \mathbb{H}(i)\phi(k,j)\delta_{k<i} \\ &= [\phi(k,i)\mathbb{H}(i)\phi(i,j)\delta_{k\geq i} - \phi(k,j)\mathbb{H}(i) + \mathbb{H}(i)\phi(k,j)\delta_{k<i}]\delta_{j<i}\end{aligned}$$

∎

Lemma 3.5 **: Sensitivity of $\boldsymbol{H}\phi$**

$$[\dot{\boldsymbol{H}}\phi] = \boldsymbol{H}\phi[\tilde{\Omega}_\delta\phi - \tilde{\Omega}]$$
$$[\boldsymbol{H}\phi]_{\theta_i} = \boldsymbol{H}\phi[\mathbb{H}^i_\delta\phi - \mathbb{H}^i] \tag{3.41}$$

Proof:

$$[\boldsymbol{H}\phi]_{\theta_i} \overset{3.37,3.39}{=} \boldsymbol{H}_{\theta_i}\phi + \boldsymbol{H}\phi_{\theta_i} = \boldsymbol{H}\phi[\mathbb{H}^i_\delta\phi - \mathbb{H}^i]$$

∎

4 Mass Matrix Related Sensitivities

We now proceed to evaluate sensitivities for the mass matrix, using its Newton-Euler factorization $\boldsymbol{H}\phi\boldsymbol{M}\phi^*\boldsymbol{H}^*$ as a point of departure. The main difference in computing mass matrix sensitivities, when compared to the sensitivities we have computed up to now, is that the mass matrix is "quadratic" in the spatial operators ϕ and $\boldsymbol{H}$. That is, the spatial operators ϕ and $\boldsymbol{H}$ each appears twice in the Newton-Euler mass matrix factorization. This property suggests that application of the basic idea of chain-rule differentiation should be sufficient to compute the mass matrix sensitivity. This is indeed what happens, as we show below.

Lemma 4.1 **: Sensitivity of $\phi\boldsymbol{M}\phi^*$**

$$[\phi\boldsymbol{M}\phi^*]_{\theta_i} = [\phi\mathbb{H}^i_\delta + \mathbb{H}^i_s]\phi\boldsymbol{M}\phi^* - \phi\boldsymbol{M}\phi^*[\mathbb{H}^i_\delta\phi^* + \mathbb{H}^i_s] \tag{4.42}$$

Proof:

$$\begin{aligned}
[\phi\boldsymbol{M}\phi^*]_{\theta_i} &= [\phi]_{\theta_i}\boldsymbol{M}\phi^* + \phi M[\phi]^*_{\theta_i} + \phi\boldsymbol{M}_{\theta_i}\phi^* \\
&= [\phi\mathbb{H}^i\tilde{\phi} - \tilde{\phi}\mathbb{H}^i_s\phi]\boldsymbol{M}\phi^* + \phi M[\phi^*\mathbb{H}^i_s\tilde{\phi}^* - \tilde{\phi}^*\mathbb{H}^i\phi^*] + \phi[\mathbb{H}^i\boldsymbol{M} - \boldsymbol{M}\mathbb{H}^i]\phi^* \\
&= [\phi\mathbb{H}^i_\delta + \mathbb{H}^i_s]\phi\boldsymbol{M}\phi^* - \phi\boldsymbol{M}\phi^*[\mathbb{H}^i_\delta\phi^* + \mathbb{H}^i_s]
\end{aligned}$$

∎

Lemma 4.2 **: Sensitivity of the Mass Matrix $\mathcal{M}_i = [\boldsymbol{H}\phi\boldsymbol{M}\phi^*\boldsymbol{H}^*]_{\theta_i}$**

$$\mathcal{M}_{\theta_i} = \boldsymbol{H}\phi[\mathbb{H}^i_\delta\phi\boldsymbol{M} - \boldsymbol{M}\phi^*\mathbb{H}^i_\delta]\phi^*\boldsymbol{H}^* \tag{4.43}$$

Proof:

$$\mathcal{M}_{\theta_i} = \boldsymbol{H}_{\theta_i}\phi\boldsymbol{M}\phi^*\boldsymbol{H}^* + \boldsymbol{H}\phi\boldsymbol{M}\phi^*\boldsymbol{H}^*_{\theta_i} + H[\phi\boldsymbol{M}\phi^*]_{\theta_i}\boldsymbol{H}^* = \boldsymbol{H}\phi[\mathbb{H}^i_\delta\phi\boldsymbol{M} - \boldsymbol{M}\phi^*\mathbb{H}^i_\delta]\phi^*\boldsymbol{H}^*$$

∎

This is one of the central results of the paper. Lemma 4.2 establishes how to compute the mass matrix sensitivity with respect to an arbitrary hinge angle θ_i, in terms of the spatial operator ϕ itself. The spatial operator ϕ appears a total of 4 times in the mass matrix sensitivity, whereas it appears only twice in the original Newton-Euler factorization $\boldsymbol{H}\phi\boldsymbol{M}\phi^*\boldsymbol{H}^*$. However, in both cases, that of the mass matrix and its sensitivity, it is the same operator ϕ that plays a central role. This means that process of computing the mass matrix sensitivity is "closed", in the sense that the same operator ϕ that appears in the mass matrix also appears in its sensitivity. The only new operator that appears in the sensitivity, and which does not show up in the mass matrix Newton-Euler factorization, is the "pick-off" operator $\mathbb{H}^i_\delta$. However, this operator is memoryless, in the sense that no spatial recursions are needed to compute it, in contrast to the operator ϕ which involves an inward (from the tip to the base) spatial recursion. It is therefore possible to observe that, aside from the presence of the relatively trivial memory-less operator $\mathbb{H}^i_\delta$, all of the operators in the mass matrix sensitivity are identical to those in the original mass matrix factorization. The main significance of this observation is that spatial recursion set up to evaluate the mass matrix can also be used as a basis to also evaluate its sensitivity coefficients. Before proceeding further, we state below without proof an alternative expression for the sensitivity of the mass matrix using articulated body inertia quantities.

Lemma 4.3 **: Alternative expression for** $\mathcal{M}_{\theta_i}$

Note that since $\phi\boldsymbol{M}\phi^*\boldsymbol{H}^* = [I + \phi\boldsymbol{K}\boldsymbol{H}]\boldsymbol{P}\phi^*\boldsymbol{H}^*$,

$$\mathcal{M}_{\theta_i} = \boldsymbol{H}\phi[\mathbb{H}^i_\delta(I + \phi\boldsymbol{K}\boldsymbol{H})\boldsymbol{P} - \boldsymbol{P}(I + \phi\boldsymbol{K}\boldsymbol{H})^*\mathbb{H}^i_\delta]\phi^*\boldsymbol{H}^*$$

∎

5 Composite Rigid Body Inertias Based Sensitivities

Now that we have developed closed-form expressions for the mass matrix sensitivity, we shift our focus slightly to look at the computational aspects. The **Composite Rigid Body Inertias** (CRB) were introduced in [Walker 82] to develop efficient algorithms for the computation of the system mass matrix. As we will see below, the CRB also play a key role in deriving expressions for the mass matrix sensitivities that are simpler to evaluate. We begin by providing some background on the composite rigid body inertias.

5.1 The Composite Rigid Body Inertias

The **composite rigid body inertia**, $\boldsymbol{R}(k)$, at the k^{th} link is the effective spatial inertia at $\mathcal{O}_k$ of the outboard links $k\ \cdots\ 1$ assuming that they form a composite rigid (augmented) body obtained by freezing hinges $(k-1)\cdots 1$. In general the value of $\boldsymbol{R}(k)$ is not a constant and rather depends on the configuration of the hinges 1 through $(k-1)$. Clearly, $\boldsymbol{R}(1) = \boldsymbol{M}(1)$, the spatial inertia of link 1. $\boldsymbol{R}(2)$ is the spatial inertia at $\mathcal{O}_2$ of links 1 and 2 regarded as a composite rigid body formed by "freezing" hinge 2 and ignoring the inboard links. Now let us examine how we might go about assembling these composite rigid body inertias for all the links. The composite rigid body inertia, $\boldsymbol{R}(k)$, at the k^{th} link can be obtained by using the parallel axis theorem to combine together the composite rigid body inertia $\boldsymbol{R}(k-1)$ at link $(k-1)$, with the spatial inertia, $\boldsymbol{M}(k)$ of the k^{th} link. This procedure gives us the tip-to-base recursive computational algorithm in Eq. 5.44 for assembling the composite rigid body inertias for all the links in the serial chain.

Recursive Computation of Composite Body Inertias

$$\left\{\begin{array}{l} \qquad\qquad \boldsymbol{R}(0) = 0 \\ \textbf{for}\ \boldsymbol{k\ =\ 1\cdots n} \\ \qquad \boldsymbol{R}(k) \quad = \phi(k,k-1)\boldsymbol{R}(k-1)\phi^*(k,k-1) + \boldsymbol{M}(k) \\ \textbf{end loop} \end{array}\right. \tag{5.44}$$

Lemma 5.1 **: Decomposition of the Mass Matrix**

We have the following decomposition of the mass matrix into diagonal and triangular factors:

$$\mathcal{M} = \boldsymbol{H}\boldsymbol{R}\boldsymbol{H}^* + \boldsymbol{H}\tilde{\phi}\boldsymbol{R}\boldsymbol{H}^* + \boldsymbol{H}\boldsymbol{R}\tilde{\phi}^*\boldsymbol{H}^* \tag{5.45}$$

Proof: *See [Jain 91].* ∎

From this decomposition it follows the the elements of $\mathcal{M}$ are given by the following expression:

$$\mathcal{M}(i,j) = \left\{\begin{array}{ll} \boldsymbol{H}(i)\boldsymbol{R}(i)\boldsymbol{H}^*(i) & \text{for } i = j \\ \boldsymbol{H}(i)\phi(i,j)\boldsymbol{R}(j)\boldsymbol{H}^*(j) & \text{for } i > j \\ \mathcal{M}^*(j,i) & \text{for } i < j \end{array}\right. \tag{5.46}$$

5.2 Inertial frame reference quantities

The hinge map $\boldsymbol{H}^*(k)$ characterizes the relative spatial velocity $\boldsymbol{H}^*(k)\dot{\theta}(k)$ across the k^{th} hinge with the hinge frame $\mathcal{O}_k$ as the velocity reference frame. However one can also choose an inertially fixed frame - denoted $\mathbb{I}$ - as the velocity reference frame. This choice helps simplify some of the analytical expressions as seen below. the corresponding hinge map $\boldsymbol{H}_{\mathbb{I}}{}^*(k)$ is given by

$$\boldsymbol{H}_{\mathbb{I}}{}^*(k) = \phi^*(k,\mathbb{I})\boldsymbol{H}^*(k) = \begin{bmatrix} h(k) \\ \lambda(k) \end{bmatrix}, \quad \lambda(k) \ \triangleq\ \tilde{l}(\mathbb{I},k)h(k) \tag{5.47}$$

$$\mathbb{H}_{\mathbb{I}}(k) \stackrel{\triangle}{=} \mathcal{S}\left[\boldsymbol{H}_{\mathbb{I}}{}^*(k)\right] = \phi(\mathbb{I},k)\mathbb{H}(k)\phi(k,\mathbb{I}) = \begin{pmatrix} \tilde{h}(k) & \tilde{\lambda}(k) \\ 0 & \tilde{h}(k) \end{pmatrix} \tag{5.48}$$

Thus, for any spatial vector X,

$$\mathbb{H}_{\mathbb{I}}(i)X = -\mathcal{Q}[X]\boldsymbol{H}_{\mathbb{I}}{}^*(i) \tag{5.49}$$

Note that the corresponding inertial frame referenced CRB inertia - denoted $\boldsymbol{R}_{\mathbb{I}}(k)$ - is given by the relation

$$\boldsymbol{R}_{\mathbb{I}}(k) = \phi(n,k)\boldsymbol{R}(k)\phi^*(n,k) \tag{5.50}$$

The following lemma provides an alternative operator expression for the mass matrix sensitivity using the composite rigid body inertia operators. These expressions are easier to evaluate as illustrated by the later results which provide expressions for the component elements of the mass matrix sensitivity.

***Lemma 5.2* : CRB Based Expression for $\mathcal{M}_{\theta_i}$**

$$\mathcal{M}_{\theta_i} = \boldsymbol{H}\Big[\phi[\mathbb{H}^i_\delta\boldsymbol{R} - \boldsymbol{R}\mathbb{H}^i_\delta]\phi^* + \phi\mathbb{H}^i_\delta\tilde{\phi}\boldsymbol{R} - \boldsymbol{R}\tilde{\phi}^*\mathbb{H}^i_\delta\phi^*\Big]\boldsymbol{H}^* \tag{5.51}$$

Proof: *The above expression follows from the use of Eq. 5.46 in Eq. 5.52.*

∎

***Lemma 5.3* : Component elements of $\mathcal{M}_{\theta_i}$**

We have that

$$\begin{aligned}
\mathcal{M}_{\theta_i}(j,k) &= \mathcal{M}_{\theta_i}(k,j) \\
&= \boldsymbol{H}(k)\Big[\phi(k,i)[\mathbb{H}(i)\boldsymbol{R}(i) - \boldsymbol{R}(i)\mathbb{H}(i)]\phi^*(j,i)\delta_{k,j>i} \\
&\qquad + \phi(k,i)\mathbb{H}(i)\phi(i,j)\boldsymbol{R}(j)\delta_{j\leq i<k} - \boldsymbol{R}(k)\phi^*(i,k)\mathbb{H}(i)\phi^*(j,i)\delta_{k\leq i<j}\Big]\boldsymbol{H}^*(j) \\
&= \boldsymbol{H}_{\mathbb{I}}(j)[\mathbb{H}_{\mathbb{I}}(i)\boldsymbol{R}_{\mathbb{I}}(k)\delta_{k\leq i<j} + \boldsymbol{R}_{\mathbb{I}}(j)\mathbb{H}^*_{\mathbb{I}}(i)\delta_{j\leq i<k} + \{\mathbb{H}_{\mathbb{I}}(i)\boldsymbol{R}_{\mathbb{I}}(i) + \boldsymbol{R}_{\mathbb{I}}(i)\mathbb{H}^*_{\mathbb{I}}(i)\}\delta_{k,j>i}]\boldsymbol{H}_{\mathbb{I}}{}^*(k) \\
&= \begin{cases} \boldsymbol{H}_{\mathbb{I}}(j)\mathbb{H}_{\mathbb{I}}(i)\boldsymbol{R}_{\mathbb{I}}(k)\boldsymbol{H}_{\mathbb{I}}{}^*(k) & \textit{for } j>i\geq k \\ 0 & \textit{for } j,k\leq i \\ \boldsymbol{H}_{\mathbb{I}}(j)\Big\{\mathbb{H}_{\mathbb{I}}(i)\boldsymbol{R}_{\mathbb{I}}(i) + \boldsymbol{R}_{\mathbb{I}}(i)\mathbb{H}^*_{\mathbb{I}}(i)\Big\}\boldsymbol{H}_{\mathbb{I}}{}^*(k) & \textit{for } k,j>i \end{cases}
\end{aligned} \tag{5.52}$$

Proof: *Note that $\mathcal{M}_{\theta_i}(j,k) = 0$ for $j,k\leq i$.*

$$\begin{aligned}
\mathcal{M}_{\theta_i}(j,k) &= \boldsymbol{H}(k)\Big\{\phi(k,i)\mathbb{H}(i)[\phi\boldsymbol{M}\phi^*](i,j)\delta_{k>i} - [\phi\boldsymbol{M}\phi^*](k,i)\mathbb{H}(i)\phi^*(j,i)\delta_{j>i}\Big\}\boldsymbol{H}^*(j) \\
&= \boldsymbol{H}(k)\Big[\phi(k,i)[\mathbb{H}(i)\boldsymbol{R}(i) - \boldsymbol{R}(i)\mathbb{H}(i)]\phi^*(j,i)\delta_{k,j>i} + \phi(k,i)\mathbb{H}(i)\phi(i,j)\boldsymbol{R}(j)\delta_{j\leq i<k} \\
&\qquad - \boldsymbol{R}(k)\phi^*(i,k)\mathbb{H}(i)\phi^*(j,i)\delta_{k\leq i<j}\Big]\boldsymbol{H}^*(j)
\end{aligned}$$

∎

This result shows an explicit expression for the sensitivity with respect to hinge θ_i of the general matrix element $\mathcal{M}(j,k)$. The key quantity involved in this expression is the composite body inertia $\boldsymbol{R}(i)$ evaluated at the same hinge i. Since both the argument i in $\boldsymbol{R}(i)$ refer to the same hinge, this means that it is possible to compute this quantity with a single spatial recursion, going in the inward direction from the tip of the serial-chain to the base. This recursive algorithm is outlined in the following result. The above result implies that once we have either of the $\boldsymbol{R}(i)$ or $\boldsymbol{R}_{\mathbb{I}}(i)$ composite rigid body inertias computed (using Eq. 5.44), it is a simple matter to obtain the individual elements of the mass matrix sensitivity matrix.

5.3 Christoffel Symbols

Christoffel symbols play a key role in multibody dynamics quantities and can be used to compute the Coriolis and gyroscopic velocity dependent acceleration terms. Recall that the Christoffel symbols are defined as

$$\mathbb{C}_i(j,k) \triangleq \frac{1}{2}\left[\frac{\partial\mathcal{M}(i,j)}{\partial\theta(k)} + \frac{\partial\mathcal{M}(i,k)}{\partial\theta(j)} - \frac{\partial\mathcal{M}(j,k)}{\partial\theta(i)}\right] \tag{5.53}$$

The **symmetric** matrix $\mathbb{C}_i$ whose elements are the Christoffel symbols is such that the Coriolis joint force elements are given by the expression

$$\mathcal{C}(i) = \dot{\theta}^*\mathbb{C}_i\dot{\theta}$$

For a tree-topology system,

$$\begin{aligned}
\mathbb{C}_i(j,k) &= \mathbb{C}_i(k,j) \quad \forall\ i,j,k \\
\mathbb{C}_i(j,k) &= -\mathbb{C}_k(j,i) \quad \forall\ j \geq k,i \\
\mathbb{C}_i(j,i) &= \mathbb{C}_i(i,j) = 0 \quad \forall\ j \geq i \\
\frac{\partial\mathcal{M}(k,j)}{\partial\theta_i} &= \mathbb{C}_j(i,k) + \mathbb{C}_k(j,i) \quad \forall\ i,j,k
\end{aligned}$$

Also $\mathbb{C}_i(j,k)$ is a function of $\theta(1)\cdots\theta(m-1)$ alone where $m \triangleq max(i,j,k)$.

Lemma 5.4 **: Expressions for Christoffel symbols using Composite Body Inertias**

For $j > k, i$, we have that

$$\begin{aligned}
\mathbb{C}_i(j,k) = \frac{1}{2}\boldsymbol{H}(j)\Big\{&\Big[\phi(j,k)[\mathbb{H}(k)\boldsymbol{R}(k) - \boldsymbol{R}(k)\mathbb{H}(k)]\phi^*(i,k)\delta_{i>k} + \phi(j,k)\mathbb{H}(k)\phi(k,i)\boldsymbol{R}(i)\delta_{i\leq k}\Big]\boldsymbol{H}^*(i) \\
&- \Big[\phi(j,i)[\mathbb{H}(i)\boldsymbol{R}(i) - \boldsymbol{R}(i)\mathbb{H}(i)]\phi^*(k,i)\delta_{k,j>i} + \phi(j,i)\mathbb{H}(i)\phi(i,k)\boldsymbol{R}(k)\delta_{k\leq i<j}\Big]\boldsymbol{H}^*(k)\Big\}
\end{aligned}$$

Proof: *From Eq. 5.53 and Lemma 5.3 it follows that*

$$\begin{aligned}
2\mathbb{C}_i(j,k) = \boldsymbol{H}(j)\Big[&\phi(j,k)[\mathbb{H}(k)\boldsymbol{R}(k) - \boldsymbol{R}(k)\mathbb{H}(k)]\phi^*(i,k)\delta_{j,i>k} + \phi(j,k)\mathbb{H}(k)\phi(k,i)\boldsymbol{R}(i)\delta_{i\le k<j} \\
&- \boldsymbol{R}(j)\phi^*(k,j)\mathbb{H}(k)\phi^*(i,k)\delta_{j\le k<i}\Big]\boldsymbol{H}^*(i) \\
+ \boldsymbol{H}(k)\Big[&\phi(k,j)[\mathbb{H}(j)\boldsymbol{R}(j) - \boldsymbol{R}(j)\mathbb{H}(j)]\phi^*(i,j)\delta_{k,i>j} + \phi(k,j)\mathbb{H}(j)\phi(j,i)\boldsymbol{R}(i)\delta_{i\le j<k} \\
&- \boldsymbol{R}(k)\phi^*(j,k)\mathbb{H}(j)\phi^*(i,j)\delta_{k\le j<i}\Big]\boldsymbol{H}^*(i) \\
- \boldsymbol{H}(k)\Big[&\phi(k,i)[\mathbb{H}(i)\boldsymbol{R}(i) - \boldsymbol{R}(i)\mathbb{H}(i)]\phi^*(j,i)\delta_{k,j>i} + \phi(k,i)\mathbb{H}(i)\phi(i,j)\boldsymbol{R}(j)\delta_{j\le i<k} \\
&- \boldsymbol{R}(k)\phi^*(i,k)\mathbb{H}(i)\phi^*(j,i)\delta_{k\le i<j}\Big]\boldsymbol{H}^*(j)
\end{aligned}$$

For $j > k, i$, we have that

$$\begin{aligned}
2\mathbb{C}_i(j,k) = \boldsymbol{H}(j)\Big[&\phi(j,k)[\mathbb{H}(k)\boldsymbol{R}(k) - \boldsymbol{R}(k)\mathbb{H}(k)]\phi^*(i,k)\delta_{i>k} + \phi(j,k)\mathbb{H}(k)\phi(k,i)\boldsymbol{R}(i)\delta_{i\le k}\Big]\boldsymbol{H}^*(i) \\
- \boldsymbol{H}(k)\Big[&\phi(k,i)[\mathbb{H}(i)\boldsymbol{R}(i) - \boldsymbol{R}(i)\mathbb{H}(i)]\phi^*(j,i)\delta_{k,j>i} - \boldsymbol{R}(k)\phi^*(i,k)\mathbb{H}(i)\phi^*(j,i)\delta_{k\le i<j}\Big]\boldsymbol{H}^*(j)
\end{aligned}$$

For $j > k > i$, we have that

$$2\mathbb{C}_i(j,k) = \boldsymbol{H}(j)\phi(j,k)\mathbb{H}(k)\phi(k,i)\boldsymbol{R}(i)\boldsymbol{H}^*(i) - \boldsymbol{H}(k)\phi(k,i)[\mathbb{H}(i)\boldsymbol{R}(i) - \boldsymbol{R}(i)\mathbb{H}(i)]\phi^*(j,i)\boldsymbol{H}^*(j)$$

■

Lemma 5.5 **: Alternative expression for the Christoffel symbols**

Define $\mathcal{Y}(a,b,c)$ as

$$\mathcal{Y}(a,b,c) = \frac{1}{2}\boldsymbol{H}_{\mathbb{I}}(a)\Big[-\mathcal{Q}[\boldsymbol{R}_{\mathbb{I}}(c)\boldsymbol{H}_{\mathbb{I}}{}^*(c)] + \mathbb{H}_{\mathbb{I}}(c)\boldsymbol{R}_{\mathbb{I}}(c) + \boldsymbol{R}_{\mathbb{I}}(c)\mathbb{H}_{\mathbb{I}}^*(c)\Big]\boldsymbol{H}_{\mathbb{I}}{}^*(b) \tag{5.54}$$

Then, an alternative expression for the Christoffel symbols is as follows:

$$\mathbb{C}_i(j,k) = \begin{cases} -\mathcal{Y}(k,j,i) & \text{for } j > k > i \\ 0 = \mathcal{Y}(i,j,i) & \text{for } j \ge k = i \\ \mathcal{Y}(i,j,k) & \text{for } j > i > k \\ \mathcal{Y}(i,i,k) & \text{for } j = i > k \\ \mathcal{Y}(i,j,k) & \text{for } i > j > k \\ \mathcal{Y}(i,k,j) & \text{for } i > k > j \end{cases} \tag{5.55}$$

Proof: *This result is obtained by combining the expressions in Section 5.2 with Lemma 5.4.* ■

6 Concluding Remarks

We have covered a range of topics all within the overarching goal of developing various expressions for the mass matrix sensitivity coefficients, with respect to arbitrarily specified joint angle changes. Spatial operators make possible the systematic development of these quantities. The Newton-Euler spatial operator factorization of the mass matrix is used as a starting point to derive the mass matrix sensitivity equations. Alternatively, the composite-body mass matrix is then used to derive alternative equations that are relatively simpler to evaluate. The sensitivity computations are then used to evaluate the velocity dependent Christoffel symbols in Lagrange's equations of motion. To our knowledge, this is the first time that this type of term has been computed explicitly, using only spatial recursions and without the need for symbolic or numerical differentiation. The paper focused only on the mass matrix and its related sensitivity. We have developed similar results for the mass matrix inverse and its sensitivity, and these results will be described in a separate publication.

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