Instanton Approach to the Langevin Motion of a Particle in a Random Potential

A. V. Lopatin[1] and V. M. Vinokur[2]
[1]*Department of Physics, Rutgers University, Piscataway, New Jersey 08854*
[2]*Material Science Division, Argonne National Laboratory, Argonne, Illinois 60439*
(Received 7 July 2000)

We develop an instanton approach to the nonequilibrium dynamics in one-dimensional random environments. The long time behavior is controlled by rare fluctuations of the disorder potential and, accordingly, by the tail of the distribution function for the time a particle needs to propagate along the system (the delay time). The proposed method allows us to find the tail of the delay time distribution function and delay time moments, providing thus an exact description of the long time dynamics. We analyze arbitrary environments covering different types of glassy dynamics: dynamics in a short-range random field, creep, and Sinai's motion.

DOI: 10.1103/PhysRevLett.86.1817 PACS numbers: 72.80.Ng, 73.61.–r, 75.10.Nr

One-dimensional driven dynamics in a random environment attracts a great deal of current attention. The motivation of the interest is twofold: First, the propagation of a particle through a 1D random potential has become a paradigm of generality for out-of-equilibrium stochastic processes in random systems capturing all effects of glassy dynamics including aging and memory effects [1–4]. Second, a particle moving in a 1D random potential models straightforwardly a variety of physical systems ranging from dislocations and charge density waves in solids, spin-chain dynamics and domain growth, to protein molecules and bacterial colonies [1,2,4–6]. The attraction of the 1D models is thus that while allowing for an analytical treatment (and often even for a full analytical solution) they also offer insight into generic basic properties of the wealth of glassy out-of-equilibrium systems. Indeed, even the simplest 1D models with the Gaussian correlated potential $v(x)$, with $\langle v(x)\rangle = 0$, $\langle [v(x) - v(0)]^2\rangle = \kappa|x|^\gamma$ exhibit a striking generality and diversity of glassy behaviors [7,8].

The approach employed in [7,8] enables the exact derivation of the particle velocity, but it does not allow for a complete dynamic description of the system—for example, in terms of velocity cumulants and/or correlation functions. Taking a kinetic view of the problem one can characterize the dynamic properties of a random system by the probability distribution for the particle velocity or, equivalently, by the probability distribution for the particle delay time τ. The latter is defined as an average time that a driven particle spends to propagate through the sample, and the corresponding distribution function $P(\tau)$ characterizes completely the transport properties of the system involved. The seminal works [9–11] demonstrated the power of the $P(\tau)$-based approach to stochastic transport in disordered solids and showed that the algebraic $P(\tau)$ leads to anomalous diffusion. Yet the mechanism for the origin of such algebraic tails or, more generally, of the Levy distributions for τ remained an open question. The aim of our work is to bridge between the solvable 1D models and the master equation methods of [9–11].

The exact expressions for a particle velocity derived in [7,8] indicate that the system dynamics is governed by rare fluctuations of the random potential, i.e., by the tail of the delay time distribution function. Thus the *instanton* solution for the Langevin equation in the presence of an external force offers a most adequate description of the long time glassy dynamics in an arbitrary 1D random environment. In this Letter we apply the instanton method developed earlier in [12,13] to find the asymptotic behavior of the delay time distribution function $P(\tau)$ at large τ. In general, the idea of the instanton method is to pick up the largest contribution to the functional integral coming from an optimal configuration instead of implementing the complete integration. In our case of low enough temperatures, the delay time is mainly determined by the transition over the largest barrier in the system. Since this time is exponentially large, one expects that the delay time averaged over the random potential (or any of its higher moments) is determined by the saddle point trajectory of some effective action controlling this exponential behavior, and, thus, it can be most appropriately found by the instanton method indeed. Further, knowing all the moments of the delay time one can reconstruct the large τ asymptotics of the distribution function $P(\tau)$.

The distribution function $P(\tau)$ depends strongly on the specific form of the correlation function of the random potential $u(x)$. We restrict ourselves to Gaussian disorder with the correlation function

$$\langle v(x_1)v(x_2)\rangle_d = u(x_1 - x_2), \qquad \langle v(x)\rangle = 0. \tag{1}$$

We use also the function

$$K(x) = \langle [v(0) - v(x)]^2\rangle_d = 2[u(0) - u(x)], \tag{2}$$

which is more convenient in the case of the long-range correlated potential. In the case of the short-range correlated potential, when $u(x)$ monotonically decreases to zero as a function of $|x|$, we find that the delay time distribution is log-normal

$$\ln\tilde{P}(Y) = -\frac{T^2Y^2}{4u(0)}, \tag{3}$$

where $Y = \ln\tau$. The distribution function $\tilde{P}(Y)$ is defined as the distribution function of $\ln\tau$, being therefore related to the distribution function $P(\tau)$ by

$$P(\tau) = \frac{1}{\tau}\tilde{P}(\ln\tau). \quad (4)$$

In the case of the long-range correlated potential $K(x) = \kappa|x|^{\gamma}$, with $\kappa > 0$ and $0 < \gamma < 1$, the distribution function essentially depends on the applied field E:

$$\ln\tilde{P}(Y) = -\frac{2}{\kappa}\left[\frac{TY}{2-\gamma}\right]^{2-\gamma}\left[\frac{E}{\gamma}\right]^{\gamma}. \quad (5)$$

And, finally, for the so-called Sinai model [14] $K(x) = \kappa|x|$ we obtain

$$P(\tau) \sim \tau^{-2TE/\kappa-1}, \quad (6)$$

which agrees with the earlier result of Ref. [3] (see also [1]).

The model and results.—The dynamics of a particle in a potential $v(x)$ is described by the Langevin equation

$$\Gamma^{-1}\partial_t x(t) = -\beta\partial_x v(x) + \beta E + \xi(t), \quad (7)$$

where β is the inverse temperature, Γ is the inverse relaxation time, $v(x)$ is the random potential, E is the applied uniform field, and $\xi(t)$ is the Langevin thermal noise that models the thermal environment,

$$\langle\xi(t_1)\xi(t_2)\rangle_T = 2\Gamma^{-1}\delta(t_1 - t_2). \quad (8)$$

To distinguish between the average over the thermal noise and disorder averaging we denote the former by the subscript T. The inverse relaxation time Γ will be set henceforth to 1 for convenience. Making use of the standard approach (see, for example, Ref. [15]) one can write the Lagrangian corresponding to Eq. (7) in the form

$$\mathcal{L} = -\frac{1}{4}[\partial_t x - \beta E + \beta\partial_x v(x)]^2 + \frac{\beta}{2}v''(x). \quad (9)$$

It is very convenient to decouple the square in the above Lagrangian introducing an auxiliary field $\hat{x}$:

$$\mathcal{L} = -\hat{x}^2 - i\hat{x}[\partial_t x - \beta E + \beta\partial_x v(x)] + \frac{\beta}{2}v''(x). \quad (10)$$

Further we will set $\beta = 1$ measuring energies in the units of temperature. The probability for a particle to go from point x_1 to x_2 is given by the functional integral

$$P(x_1, x_2) = \int D[x(t)]D[\hat{x}(t)]\,e^{\int dt\,\mathcal{L}}, \quad (11)$$

and the time a particle spends moving from x_1 to x_2 (the delay time) is, correspondingly,

$$\tau(x_1, x_2) \sim P^{-1}(x_1, x_2). \quad (12)$$

If the temperature is lower than a typical barrier, one can use the saddle point approximation to find P:

$$P(x_1, x_2) = e^{-\mathcal{A}_{\text{s.p.}}}, \quad (13)$$

where the action $A_{\text{s.p.}}$ is the saddle point value of the action A corresponding to the Lagrangian (10): $\mathcal{A} = -\int dt\,\mathcal{L}$. Within the accuracy of the saddle point approximation the delay time $\tau(x_1, x_2)$ is given by the saddle point value of the action $\mathcal{A}$:

$$\tau(x_1, x_2) \sim P^{-1}(x_1, x_2) = e^{\mathcal{A}_{\text{s.p.}}}. \quad (14)$$

The delay time $\tau(x_1, x_2)$ averaged over the disorder is

$$\langle\tau(x_1, x_2)\rangle_d = \int D[v]\,e^{A[v,x]}, \quad (15)$$

where the effective action

$$A[v,x] = -\frac{1}{2}\int dx_1 dx_2\, v(x_1)f(x_1,x_2)v(x_2) - \int dt\,\mathcal{L} \quad (16)$$

is to be taken at the saddle configuration with respect to fields $x, \hat{x}$. The function f in (16) is the inverse correlation function (1): $\int dx\, f(x_1,x)u(x,x_2) = \delta(x_1 - x_2)$. Taking the variational derivatives of the action (16) with respect to $\hat{x}$, x, v we find the saddle point equations

$$-2\hat{x} + \partial_t x - E + \partial_x v(x) = 0, \quad (17)$$

$$\partial_t\hat{x} - \hat{x}v''(x) = 0, \quad (18)$$

$$v(x) = -\int dt\,\hat{x}(t)u'[x - x(t)], \quad (19)$$

where the variable $\hat{x}$ was redefined $i\hat{x} \to \hat{x}$, so that the saddle value of the redefined variable $\hat{x}$ became real. The fact that the original variable $\hat{x}$ has imaginary saddle value is not a problem since the contour of integration over $\hat{x}$ may be shifted parallel to the imaginary axis in the complex plain. When deriving Eqs. (17)–(19) the last term in Eq. (10) was neglected because it gives the $\mathcal{O}(1)$ contribution to the action while the whole action is much larger than one. Following Ref. [12] (see also Ref. [13]), in order to find an instanton solution we choose $\hat{x} = \partial_t x$. Indeed, this allows one to reduce Eqs. (17) and (18) to a single equation:

$$\partial_t x = -E + \partial_x v(x). \quad (20)$$

Assuming that the instanton solution $\hat{x} = \partial_t x$ exists in the interval (η_1, η_2) we find from Eq. (19)

$$v(x) = u(x - \eta_2) - u(x - \eta_1). \quad (21)$$

The interval (η_1, η_2), where the instanton solution exists, must lie within the interval (x_1, x_2); we will see later, however, that these intervals do not necessarily coincide. Outside the interval (η_1, η_2) one should take the normal solution in the form

$$\hat{x} = 0, \qquad \partial_t x = E - \partial_x v(x). \quad (22)$$

The action (16) corresponding to the solution (20) is

$$A = \frac{1}{2}[v(\eta_2) - v(\eta_1)] - E(\eta_2 - \eta_1). \quad (23)$$

Now we need to find an equation that determines the boundaries of the instanton solution. We assume that the disorder correlation function $u(x)$ is smooth enough, namely, that the second derivative is finite. Then it follows from Eqs. (18) and (21) that $\hat{x}$ is continuous because otherwise the first term in Eq. (18) would have been singular while the second term according to (21) remained regular. Since within the interval $(\eta_1, \eta_2)\hat{x} = \partial_t x$, and $\hat{x} = 0$ outside it, then $\hat{x} = \partial_t x = 0$ at the boundaries η_1, η_2, and from Eqs. (20) and (21) we get the equation

$$E = -u'(\eta_2 - \eta_1) \quad (24)$$

defining $\eta_2 - \eta_1$, and the action (23) becomes

$$A = u(0) - u(\eta) - E\eta, \quad (25)$$

where $\eta = \eta_2 - \eta_1$. Note that Eq. (24) corresponds to the extremum of the action (25) with respect to η, so the final expression can be presented as the maximum of

$$A = \frac{1}{2}K(\eta) - E\eta, \quad (26)$$

with $K(x)$ given by (2). In case of long-range potential $K(x) \sim |x|^\gamma$ the above considerations will still hold if we regularize the function $K(x)$ using, for example, regularization $|x| \to (x^2 + \epsilon)^{1/2}$ with ϵ being a small positive number. The necessity of such regularization is physically natural since the correlation function $K(x)$ must be smooth. Thus, formula (26) applies to an arbitrary disorder correlation function, its concrete realization following from the specific form of $K(x)$.

(i) Short-range potential: We begin with the simplest case of $u(x) \to 0$ monotonically with growing $|x|$. The corresponding form of the effective potential $v(x)$ is shown in Fig. 1. From Eq. (24) it follows that $\eta \to \infty$ when $E \to 0$; thus if E is low enough, the action becomes

$$A = u(0). \quad (27)$$

(ii) Potential with long-range correlations: Take the potential with correlator $K(x) = \kappa|x|^\gamma$, with $\kappa > 0$ and $0 < \gamma < 1$. In this case the expression for the action (26) still holds; however, minimizing (26) with respect to η we arrive at the action diverging at low fields:

$$A = (1 - \gamma)\left(\frac{\kappa}{2}\right)^{\frac{1}{1-\gamma}}\left(\frac{\gamma}{E}\right)^{\frac{\gamma}{1-\gamma}} \quad (28)$$

in agreement with the exact solution obtained in Ref. [8].

(iii) "Extremely correlated" disorder, $\gamma = 1$. This is the well-known Sinai model $K(x) = \kappa|x|$. Keeping in mind the regularization described above, the boundaries of the instanton solution are still determined by Eq. (24). But in this case Eq. (24) is either never satisfied or it is satisfied identically for a special value of the applied field $E = E_0 = \kappa/2$. Thus, for $E < E_0$, there exists the only instanton solution with boundaries (η_1, η_2) coinciding with the sample boundaries (x_1, x_2). The action corresponding to this solution is

$$A = (\kappa/2 - E)(x_2 - x_1), \quad (29)$$

and the average delay time essentially (exponentially) depends on the sample length. In the case $E > E_0$ the instanton solution does not exist.

Distribution function of the delay time.—The method described above can be generalized straightforwardly for a calculation of higher moments of $\tau(x_1, x_2)$:

$$\tau_n(x_1, x_2) = \langle \tau^n(x_1, x_2)\rangle_d. \quad (30)$$

The action A_n that determines the nth moment

$$\langle \tau^n(x_1, x_2)\rangle_d = e^{A_n[v,x]} \quad (31)$$

is given by

$$A_n[v, x] = -\frac{1}{2}\int dx_1 dx_2\, v(x_1) f(x_1, x_2) v(x_2) - n\int dt\, \mathcal{L}. \quad (32)$$

From this equation one can see that $A_n[u(x), x] = nA[nu(x), x]$, and from Eq. (26) we get the action A_n:

$$A_n = n^2 K(\eta)/2 - nE\eta, \quad (33)$$

which must be maximized with respect to η. This gives

$$nK'(\eta)/2 - E = 0. \quad (34)$$

Knowing all the moments of τ one can find the distribution function $P(\tau)$. Indeed, in terms of the distribution function $P(\tau)$, the moments τ_n are defined as

$$\tau_n = \int P(\tau)\tau^n\, d\tau = \int \tilde{P}(Y) e^{nY}\, dY. \quad (35)$$

Using Eq. (33) and taking integrals in Eq. (35) in the saddle point approximation we get

$$\{\ln \tilde{P}(Y) + nY\}_Y = \{n^2 K(\eta)/2 - En\eta\}_\eta, \quad (36)$$

where $\{\}_Y$ and $\{\}_\eta$ mean taking extrema with respect to Y and η, respectively. Differentiating (36) with respect to n we get

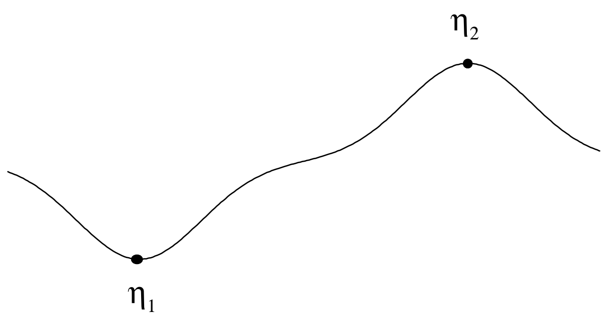

FIG. 1. The instanton solution for the potential $v(x)$ in the case of short-range potential.

$$n = \frac{Y + E\eta}{K(\eta)}. \quad (37)$$

Using Eq. (36) we find the distribution function

$$\ln\tilde{P}(Y) = -\frac{1}{2}\frac{(Y + E\eta)^2}{K(\eta)}, \quad (38)$$

where η is defined by the equation

$$K'(\eta)(Y + E\eta) - 2EK(\eta) = 0, \quad (39)$$

following from Eqs. (34) and (37). Interestingly, this equation also follows from the extremum of the logarithm of the distribution function (38) with respect to η.

The distribution function (38) is our main general result. Now we analyze different cases:

(i) For the short-range potential and not too strong applied field, Eq. (38) gives the log-normal distribution:

$$\ln\tilde{P}(Y) = -\frac{Y^2}{4u(0)}. \quad (40)$$

(ii) For the correlated potential, $K(x) = \kappa|x|^\gamma$, $0 < \gamma < 1$, Eq. (39) yields

$$\eta = \frac{\gamma Y}{E(2 - \gamma)}, \quad (41)$$

and the distribution function (38) becomes

$$\ln\tilde{P}(Y) = -\frac{2}{\kappa}\left[\frac{Y}{2 - \gamma}\right]^{2-\gamma}\left[\frac{E}{\gamma}\right]^\gamma. \quad (42)$$

(iii) Although in the Sinai case ($\gamma = 1$) the moments of τ are not defined in the limit of a large system, Eq. (5) has no singularities when $\gamma = 1$. Therefore, the distribution function in the Sinai case is given by

$$P(t) = t^{-2E/\kappa - 1}, \quad (43)$$

which agrees with the earlier result [1,3].

Discussion.—In case of disorder potential with short-range correlations the distribution function of the delay time is log-normal [see Eq. (3)] and is not sensitive to the applied electric field. Since the logarithm of the delay time is proportional to the largest barrier in the sample, the result (3) means that the distribution of the heights of the largest barrier in the sample is Gaussian. In case of the long-range correlated potential, the distribution function (5) depends essentially on the applied driving field E even in the limit of $E \to 0$. The distribution function is still normalizable and has finite moments as long as $E > 0$. The average delay time is given by (28) and it diverges exponentially when $E \to 0$; this behavior corresponds to and characterizes the creep regime. In the Sinai case the distribution function is normalizable when $E > 0$, but the moments are not defined in the limit of the large length of the sample. The average delay time shows exponential dependence on the sample size (29).

In conclusion, we have found the asymptotic behavior of the delay time distribution function for a general problem of the Langevin motion of a particle in a one-dimensional random potential. The application of our procedure to the Sinai model recovers the earlier results verifying our approach. The developed instanton method allows one to derive the distribution function for an arbitrary 1D random potential and can serve as an initial step towards a quantitative study of the glassy dynamics of the general multidimensional systems. The method proposed can also be generalized for calculation of other correlation functions which are determined by the contribution from the largest barrier in the system.

A. V. L. thanks Lev Ioffe for very useful discussions. This work was supported by the U.S. Department of Energy, Office of Science under Contract No. W-31-109-ENG-38.

[1] J.-P. Bouchaud and Antoine Georges, Phys. Rep. **195**, 127 (1990).

[2] V. M. Vinokur, J. Phys. (Paris) **47**, 1425 (1986).

[3] M. V. Feigel'man and V. M. Vinokur, J. Phys. (Paris) **49**, 1731 (1989).

[4] D. S. Fisher, P. Le Doussal, and C. Monthus, Phys. Rev. Lett. **80**, 3539 (1998).

[5] P. L. Krapivsky and E. Ben-Naim, Phys. Rev. E **56**, 3788 (1997).

[6] I. Aranson, L. Tsimring, and V. Vinokur, Phys. Rev. Lett. **79**, 3298 (1997).

[7] S. Scheidl, Z. Phys. B **97**, 345 (1995).

[8] P. Le Doussal and V. M. Vinokur, Physica (Amsterdam) **254C**, 63 (1995).

[9] G. Pfister and H. Scher, Adv. Phys. **27**, 747 (1978); H. Scher and E. W. Montroll, Phys. Rev. B **12**, 2455 (1975); J. Drager and J. Klafter, Phys. Rev. Lett. **84**, 5998 (2000).

[10] R. Metzler, E. Barkai, and J. Klafter, Phys. Rev. Lett. **82**, 3563 (1999); R. Metzler and J. Klafter, Phys. Rev. E **61**, 6308 (2000).

[11] E. W. Montroll and J. T. Bendler, J. Stat. Phys. **34**, 129 (1984).

[12] A. D. Wentzel and M. I. Friedlin, Russ. Math. Surveys **25**, 1–55 (1970); A. D. Wentzel and M. I. Friedlin, *Random Perturbations of Dynamical Systems* (Springer, New York, 1998).

[13] A. V. Lopatin and L. B. Ioffe, Phys. Rev. B **60**, 6412 (1999); L. B. Ioffe and D. Sherrington, Phys. Rev. B **57**, 7666 (1998).

[14] Y. G. Sinai, Theor. Probab. Appl. **27**, 247 (1982).

[15] K. H. Fischer and J. A. Hertz, *Spin Glasses* (Cambridge University Press, Cambridge, England, 1993).