Rigorous bounds on performance of lossy interferometers and optimal probe states

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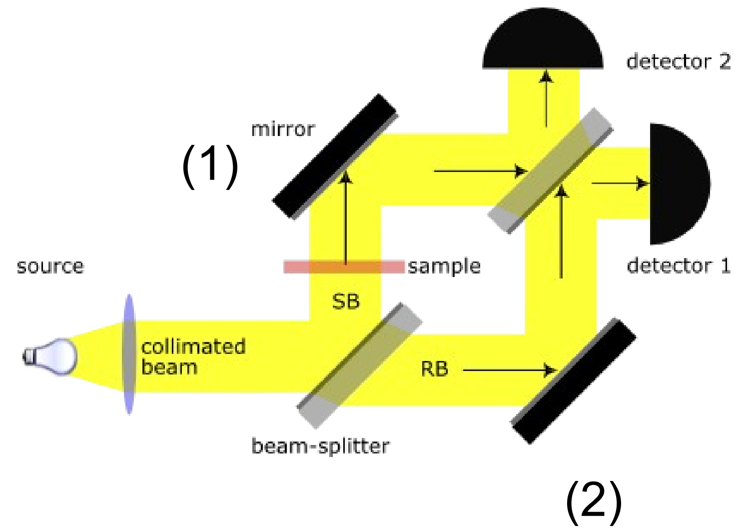
Beyond shot-noise limit

Measure small differences in the optical length

Error: $\Delta L = \frac{\lambda}{\sqrt{N}}$ (coherent light, single photons)



- Decrease wavelength
- Increase photon flux
- Use non-classical light

Single photon:

After the beamsplitter:

$$\frac{1}{\sqrt{2}}\left(|1\rangle_1|0\rangle_2+|0\rangle_1|1\rangle_2\right)$$

Pick up phase:

$$\frac{1}{\sqrt{2}}\left(e^{i\phi}|1\rangle_1|0\rangle_2+|0\rangle_1|1\rangle_2\right)$$

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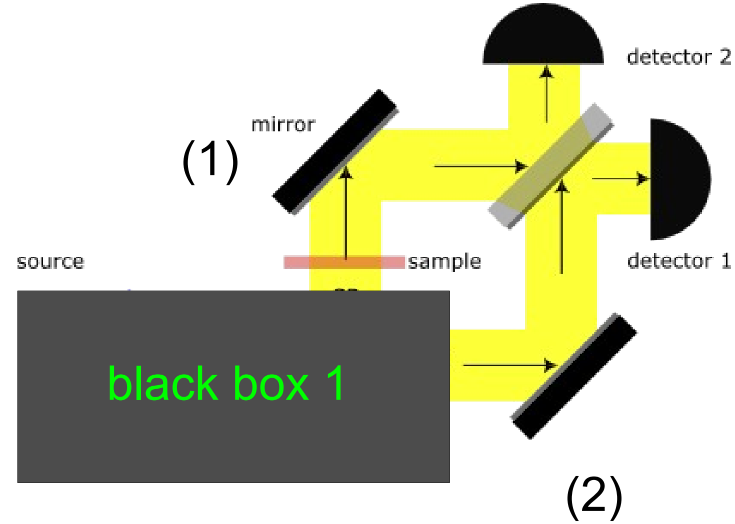


Single photon:

After the beamsplitter:

$$\frac{1}{\sqrt{2}}\left(|1\rangle_1|0\rangle_2+|0\rangle_1|1\rangle_2\right)$$

Pick up phase:

$$\frac{1}{\sqrt{2}}\left(e^{i\phi}|1\rangle_1|0\rangle_2+|0\rangle_1|1\rangle_2\right)$$

NOON state:

$$\frac{1}{\sqrt{2}}\left(|N\rangle_1|0\rangle_2+|0\rangle_1|N\rangle_2\right)$$

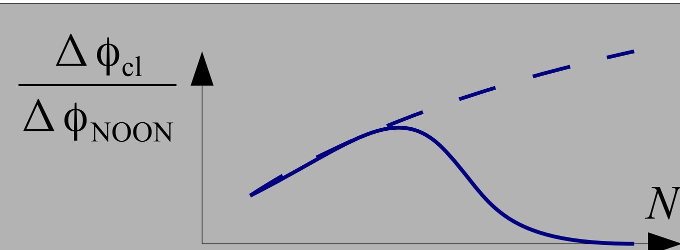
Pick up phase:

$$\frac{1}{\sqrt{2}}\left(e^{iN\phi}|N\rangle_1|0\rangle_2+|0\rangle_1|N\rangle_2\right)$$

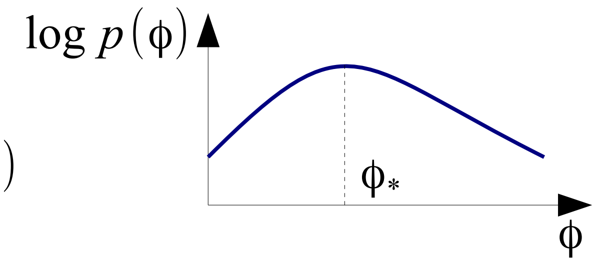
$\sqrt{N}$ precision increase (same photon number)

Limitless increase in precision?

Problem: loss is amplified $R \rightarrow R^N$



What is measurement precision?



Estimate unknown parameter (phase) φ.

Result of measurement is $k \in \{1, 2, \ldots, n\}$ with probability $p_k(\phi)$

Repeat measurements ν times; result *k* observed ν_k times: $\nu_1 + \nu_2 + \cdots + \nu_n = \nu$

$$p(\phi | \nu_1, \ldots, \nu_n) \propto \exp\left(\sum_k \nu_k \log p_k(\phi)\right) \propto \exp\left(\frac{-(\phi - \phi_*)^2}{2} \sum_k \nu_k \frac{\partial^2}{\partial \phi^2}(-\log p_k(\phi))\right)$$

ϕ_* maximizes likelihood.

$$\Delta\phi = \frac{1}{\sqrt{\nu F}}$$

$$F(\phi) = \sum_k p_k(\phi) \frac{\partial^2}{\partial \phi^2}(-\log p_k(\phi))$$ is Fisher information

Example (single photon):

$$p_{0,1}(\phi) = \frac{1 \pm \cos\phi}{2}$$

$$F(\phi) = \frac{1 + \cos\phi}{2} \cdot \frac{1}{2\cos^2\phi/2} + \frac{1 - \cos\phi}{2} \cdot \frac{1}{2\sin^2\phi/2} = 1$$

Quantum Fisher Information

(Braunstein & Caves, PRL, 1994)
(Braunstein, Caves & Milburn, Annals of Physics, 1996)

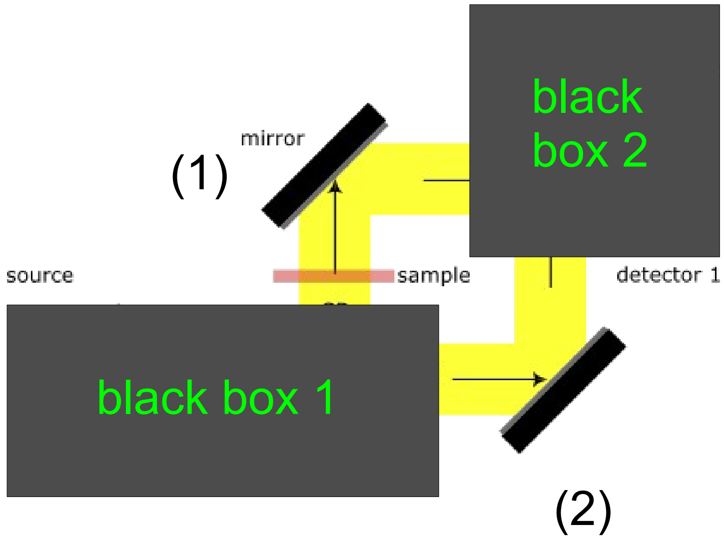
Fisher information in the optimal measurement basis = Quantum Fisher Information

Generalized uncertainty relations: $\Psi(t) = e^{-i\hat{H}t}|\Psi_0\rangle$

$$F = 4\langle\Psi_0|\hat{H}^2|\Psi_0\rangle - 4\langle\Psi_0|\hat{H}|\Psi_0\rangle^2$$

$$\langle(\Delta\hat{H})^2\rangle\langle(\Delta t)^2\rangle \geq \frac{1}{4}$$

(pure state; mixed state is more complicated)



Interferometer: $t \sim \phi$ $\hat{H} \sim \hat{n}_1$

$F_{\text{NOON}} = N^2$ (lossless case)

Photon Loss: “Fixing” NOON state

Loss of a single photon destroys entanglement:

$$\frac{1}{\sqrt{2}}\left(|N\rangle_1|0\rangle_2+|0\rangle_1|N\rangle_2\right)\rightarrow|N-1\rangle_1|0\rangle_2$$

M&M state protects against loss of up to *M* photons: (Huver, Wildfeuer, Dowling, PRA, 2008)

$$\frac{1}{\sqrt{2}}\left(|N-M\rangle_1|M\rangle_2+|M\rangle_1|N-M\rangle_2\right)$$

Numerical optimization of the probe state: (Dorner et al., PRL, 2009) (Demkowicz-Dobrzanski et al., PRA 2009)

Loss in one arm: $\alpha|N\rangle_1|0\rangle_2+\beta|M\rangle_1|N-M\rangle_2$ is nearly optimal (not too large losses)

Loss in both arms: complicated many-component state

(approximated by HB state ?)

Quantum “improvement factor” grows slower than $\sqrt{N}$

(levels off for large *N*?)

Rigorous Upper Bound

Most general *N*-photon probe state:

$$|\Phi\rangle = \sum_n \phi_n |n\rangle_1 |N-n\rangle_2$$

decays into a mixture

$$\hat{\rho} = \sum_k w_k |\Psi_k\rangle\langle\Psi_k|$$

$$|\Psi_k\rangle \propto \sum_n \sqrt{\binom{n}{k} R^k (1-R)^{n-k}}\, \phi_n |n-k\rangle_1 |N-n\rangle_2$$

k = number of lost photons

R = probability of single photon loss

$$F = \sum_k w_k F_k$$

Inequality

$$\langle\Psi_k|(\Delta \hat{n}_1)^2|\Psi_k\rangle \le \left\langle \Psi_k \left| \left(\hat{n}_1 - \frac{k}{R}\right)^2 \right| \Psi_k \right\rangle$$

Fisher information

$$F \le \sum_k \sum_n \left(n - \frac{k}{R}\right)^2 \cdot \binom{n}{k} R^k (1-R)^{n-k} \cdot |\phi_n|^2$$

$$\le \frac{4(1-R)}{R} \sum_n n |\phi_n|^2 \le \frac{4(1-R)}{R} N$$

Linear in *N*

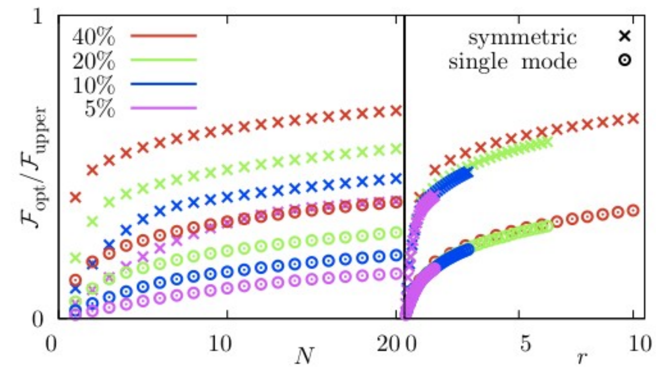
Optimal Probe States

Limit N>>1

Scaling solution:

$$F(N,R)=N^2\tilde{F}\left(\frac{NR}{1-R}\right)$$

loss parameter $r=\frac{NR}{1-R}$



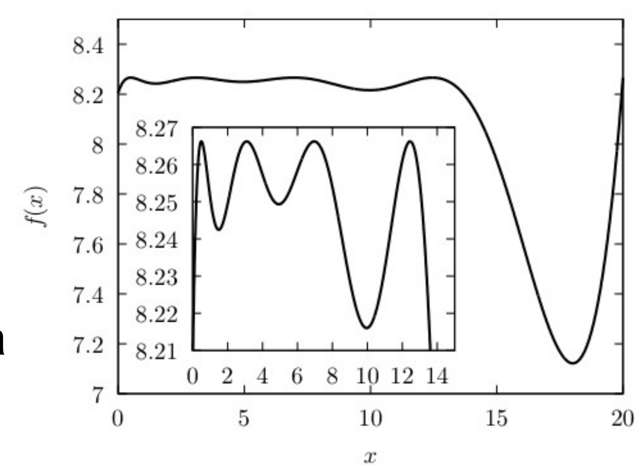
Approximate discrete set $\{\phi_n\}$ by continuous wavefunction $\phi(x)$

$$0,1,\ldots,N\rightarrow[0;r]$$

Variational calculus: $\rho(x)=|\phi(x)^2|$

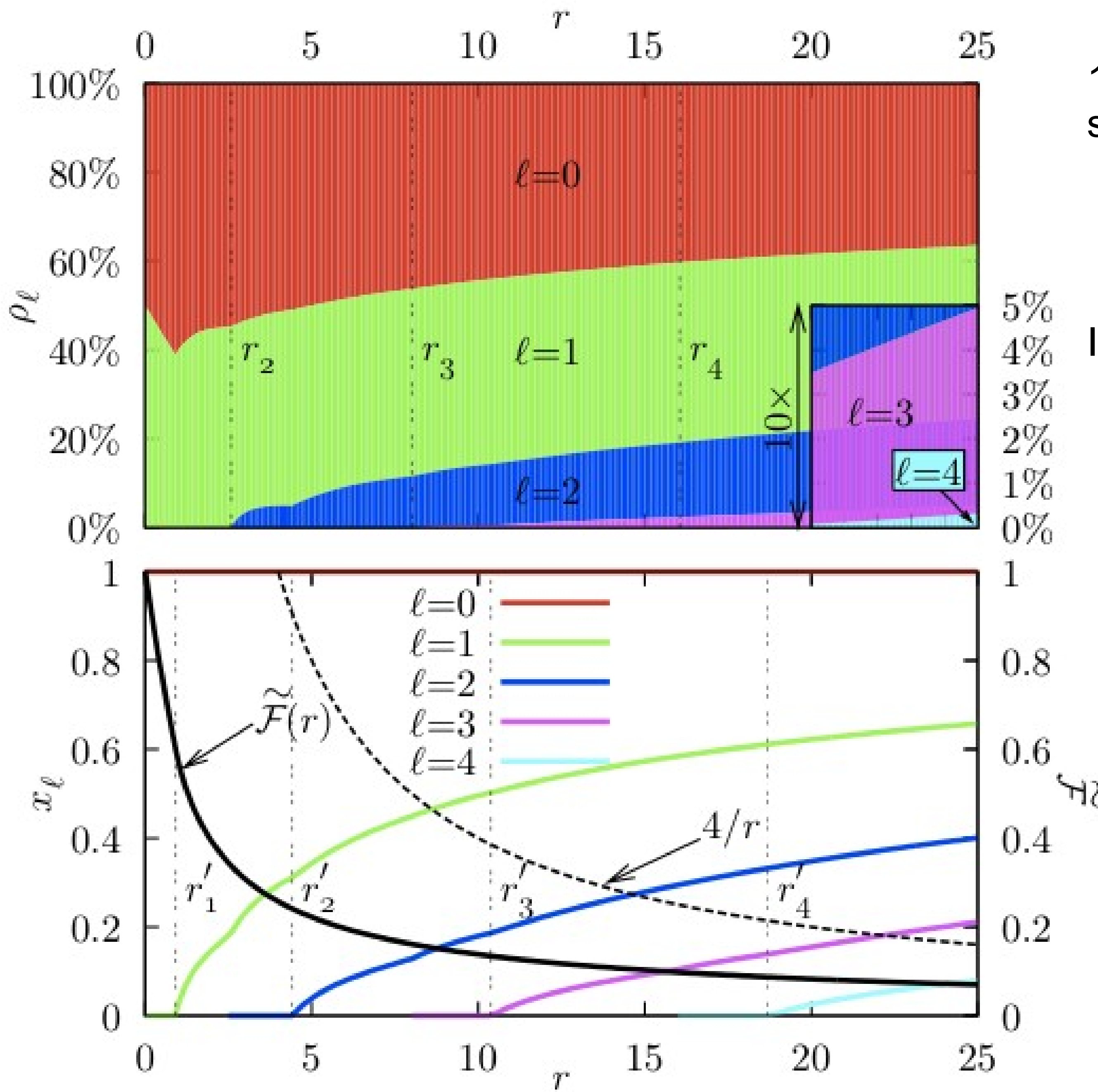
$$\Pi(x)=\frac{\delta F[\rho(x)]}{\delta\rho(x)}\leq\lambda \qquad x\in[0;r]$$

$$\rho(x)\neq 0 \text{ iff } \Pi(x)=\lambda$$



$\Pi(x)$ is entire and non-constant => finitely many maxima

Sequence of Bifurcations



1st bifurcation: component separates from the origin.

$$\sqrt{1+r_1'^2}-(1-r_1')=2\,e^{-r_1'/2}$$

Infinite sequence of bifurcations:

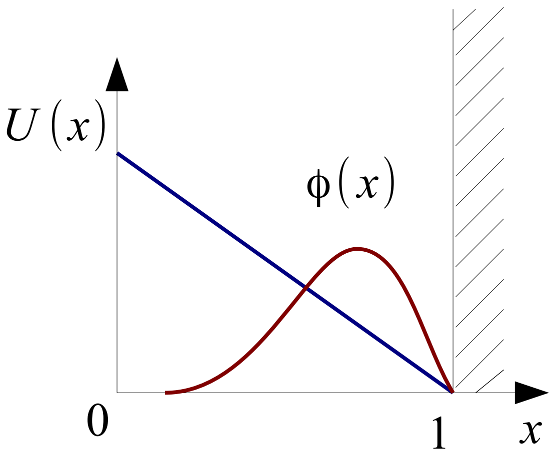
$$
\begin{aligned}
r_1' &= 0.91295692\ldots\\
r_2 &= 2.58426584\ldots\\
r_2' &= 4.40600960\ldots\\
r_3 &= 8.01095805\ldots\\
r_3' &= 10.3767442\ldots\\
r_4 &= 16.06326855\ldots\\
r_4' &= 18.69997097\ldots\\
r_5 &= 26.61426895\ldots\\
r_5' &= 29.33116503\ldots\\
r_6 &= 39.58033810\ldots\\
r_6' &= 42.26049238\ldots\\
\ldots &\;\ldots\;\ldots\;\ldots\;\ldots\;\ldots\;\ldots
\end{aligned}
$$

Fixed Loss, Large N

$r \propto N$

Formally, r is infinite. $\{\phi_n\} \to \phi(x) \qquad x \in [0;1]$

$$\sum_w w_k \langle \Psi_k | (\Delta \hat{n}_1)^2 | \Psi_k \rangle = \underbrace{\sum_k \left\langle \Psi_k \left| \left(\hat{n}_1 - \frac{k}{R} \right)^2 \right| \Psi_k \right\rangle}_{F_{\text{upper}} - \int U(x) |\phi(x)|^2 dx} - \underbrace{\sum_k \left\langle \Psi_k \left| \left(\hat{n}_1 - \frac{k}{R} \right) \right| \Psi_k \right\rangle^2}_{= \frac{m_*}{2} \int |d\phi(x)/dx|^2 \, dx}$$



$$\phi(x) = \frac{(r/4)^{1/6}}{\text{Ai}'(\mu_1)} \text{Ai} \left(\left(\frac{r}{4} \right)^{1/3} (1-x) + \mu_1 \right)$$

$$\mu_1 \approx -2.338107\ldots$$

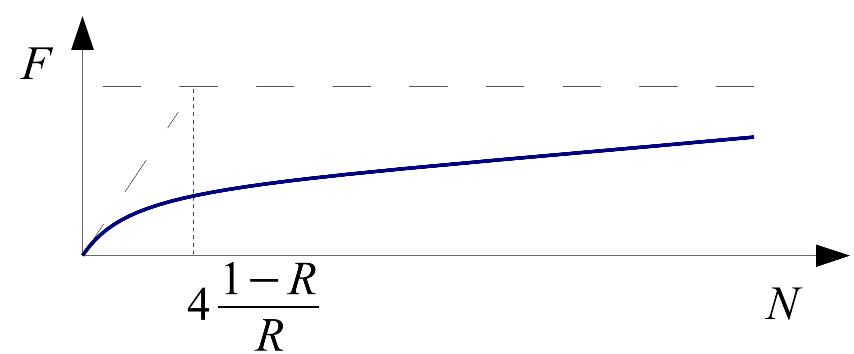
$$F = \frac{4N^2}{r} \left[1 - |\mu_1| \left(\frac{4}{r} \right)^{1/3} + \cdots \right]$$

Estimate crossover:

$\sim 0.5 \, F_{\max}$

$$r_c \sim 4 |\mu_1|^3 \sim 50$$

$$N_c \sim 50 \frac{1-R}{R}$$



Loss in Both Channels

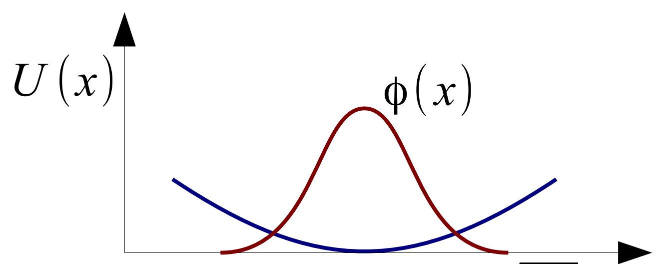
$$|\Phi\rangle\langle\Phi| \;\to\; \hat{\rho} = \sum_{k_1,k_2} w_{k_1,k_2} |\Psi_{k_1,k_2}\rangle\langle\Psi_{k_1,k_2}|$$

not an eigenvalue decomposition

Upper bound, not equality $F \leq \sum_{k_1,k_2} w_{k_1,k_2} F_{k_1,k_2}$

To find QFI: diagonalize $\hat{\rho}$, apply Braunstein-Caves formula. (difficult)

Warm-up exercise: optimize incorrect upper bound.



$$x_* = \frac{\sqrt{r^{(1)}}}{\sqrt{r^{(1)}} + \sqrt{r^{(2)}}}$$

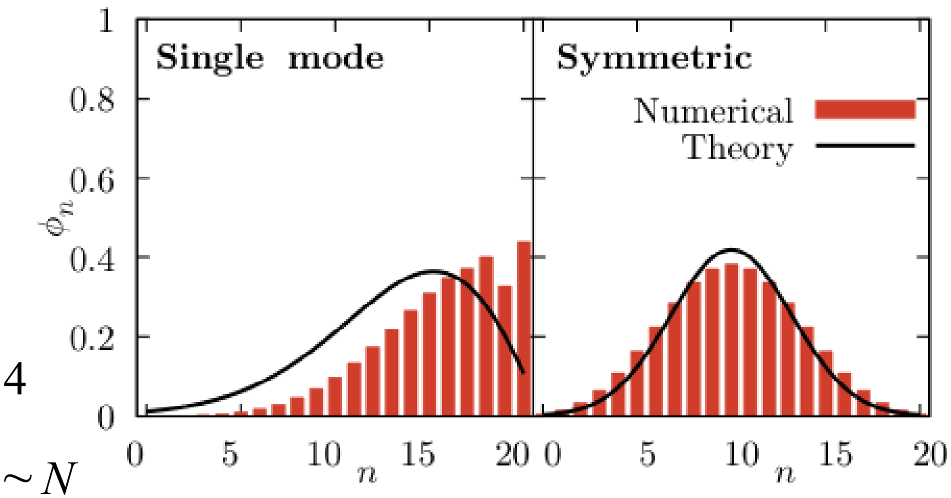
$$F = \frac{4N^2}{\left(\sqrt{r^{(1)}} + \sqrt{r^{(2)}}\right)^2} \left[1 - \frac{2}{\left(r^{(1)} r^{(2)}\right)^{1/4}} + \cdots \right]$$

- Probe state is Gaussian
- Binomial coefficients are asymptotically Gaussian
- Density matrix is Gaussian: easily diagonalizable (eigenstates of harmonic oscillator)

Surprising result: upper bound is exact. (small corrections for finite *N*)

Conclusion

- Comparison with numerics (95% loss):



Implementation?

Wavepacket width: $\sim N^{\epsilon}$ $\epsilon = 2/3$ or $\epsilon = 3/4$

All(?) known states have width $\sim\sqrt{N}$ or $\sim N$

- "Benchmark" for families of states:
  QFI per photon in the limit N>>1 as a fraction of upper bound

  - Two-component state $\alpha|N\rangle_1|0\rangle_2 + \beta|M\rangle_1|N-M\rangle_2$ 47.95%
  - Holland-Burnett state ($|N/2\rangle_1|N/2\rangle_2$+ beamsplitter) 50%

    (unequal photon numbers) <54.25%

- Improvement over classical light:

$$\frac{1}{\sqrt{R}} \quad (R_1 = R_2 = R) \qquad\qquad \frac{1+\sqrt{1-R}}{\sqrt{R}} \quad (R_1 = R,\, R_2 = 0)$$