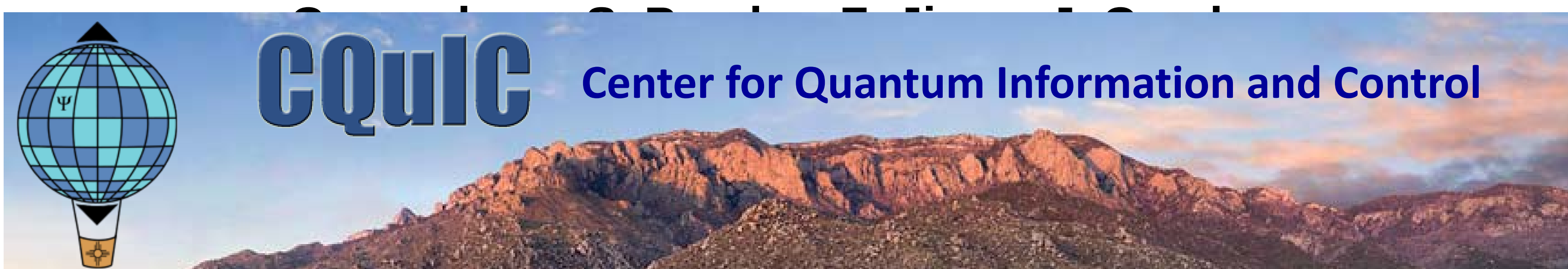
Quantum limits on linear amplifiers

I. What's the problem?
II. Quantum limits on noise in phase-preserving linear amplifiers. The whole story
III. Completely positive maps and physical ancilla states
or
IV. Nondeterministic linear amplifiers

Carlton M. Caves
Center for Quantum Information and Control, University of New Mexico
Centre for Engineered Quantum Systems, University of Queensland
http://info.phys.unm.edu/~caves



I. What's the problem?

View from Cape Hauy
Tasman Peninsula
Tasmania

Phase-preserving linear amplifiers

$$a_{\text{out}} = g a_{\text{in}} + L^\dagger$$

$$[a, a^\dagger] = 1 \quad \Longrightarrow \quad [L, L^\dagger] = g^2 - 1$$

output noise	gain	input noise	added noise

$$\langle|\Delta a_{\text{out}}|^2\rangle = g^2 \langle|\Delta a_{\text{in}}|^2\rangle + \langle|\Delta L|^2\rangle$$

$$\geq g^2 - \frac{1}{2} \qquad \geq \frac{1}{2} \qquad \geq \frac{1}{2}(g^2 - 1)$$

Refer noise to input

$$\geq 1 - \frac{1}{2g^2} \qquad \geq \frac{1}{2}\left(1 - \frac{1}{g^2}\right)$$

Added noise number $\frac{1}{2}$

$g^2 \gg 1$

Noise temperature $kT_n = \frac{\hbar\omega}{\ln 3}$

Ideal phase-preserving linear amplifier: parametric amplifier

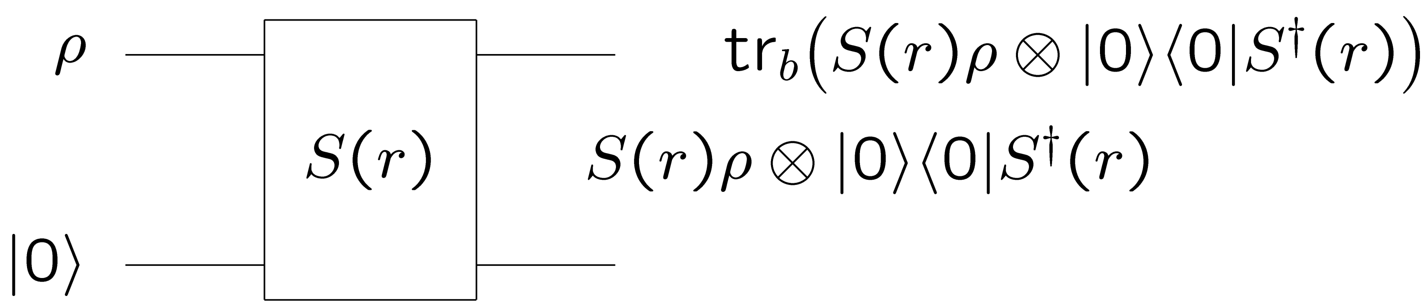
$$H = i\hbar\kappa(ab - a^\dagger b^\dagger)$$

$$\Longleftrightarrow \quad U(t,0) = e^{\kappa t(ab - a^\dagger b^\dagger)} \equiv S(r)\,, \quad r = \kappa t$$

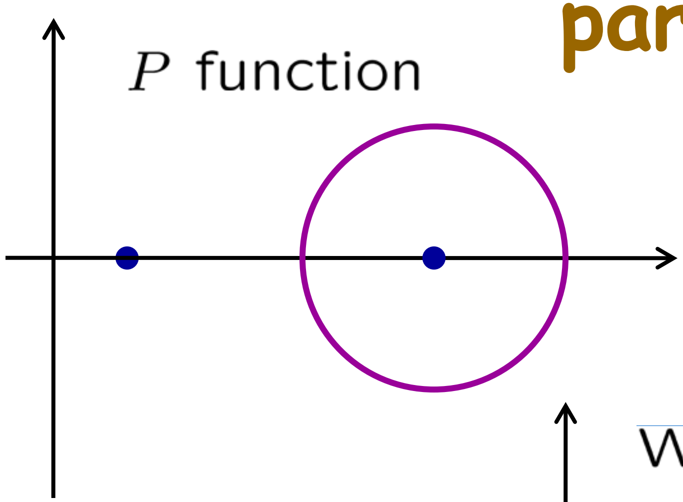
$$a_{\text{out}} = a_{\text{in}} \cosh r + b_{\text{in}}^\dagger \sinh r$$

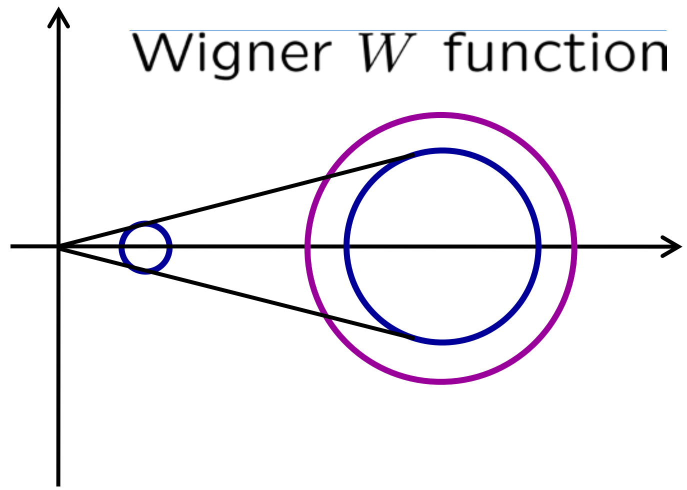
$$\cosh r = g \qquad \sinh r = \sqrt{g^2 - 1}$$

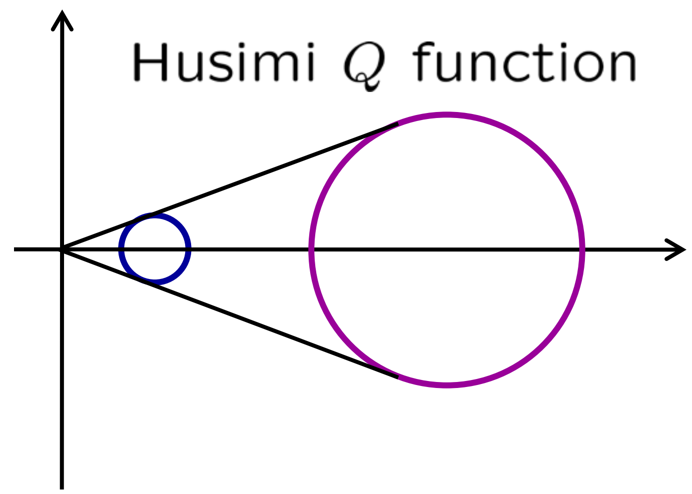
$$L = b_{\text{in}}\sqrt{g^2 - 1}$$



Ideal phase-preserving linear amplifier: parametric amplifier







The noise is Gaussian. Circles are drawn at half the standard deviation of the Gaussian.

Phase-preserving linear amplifiers

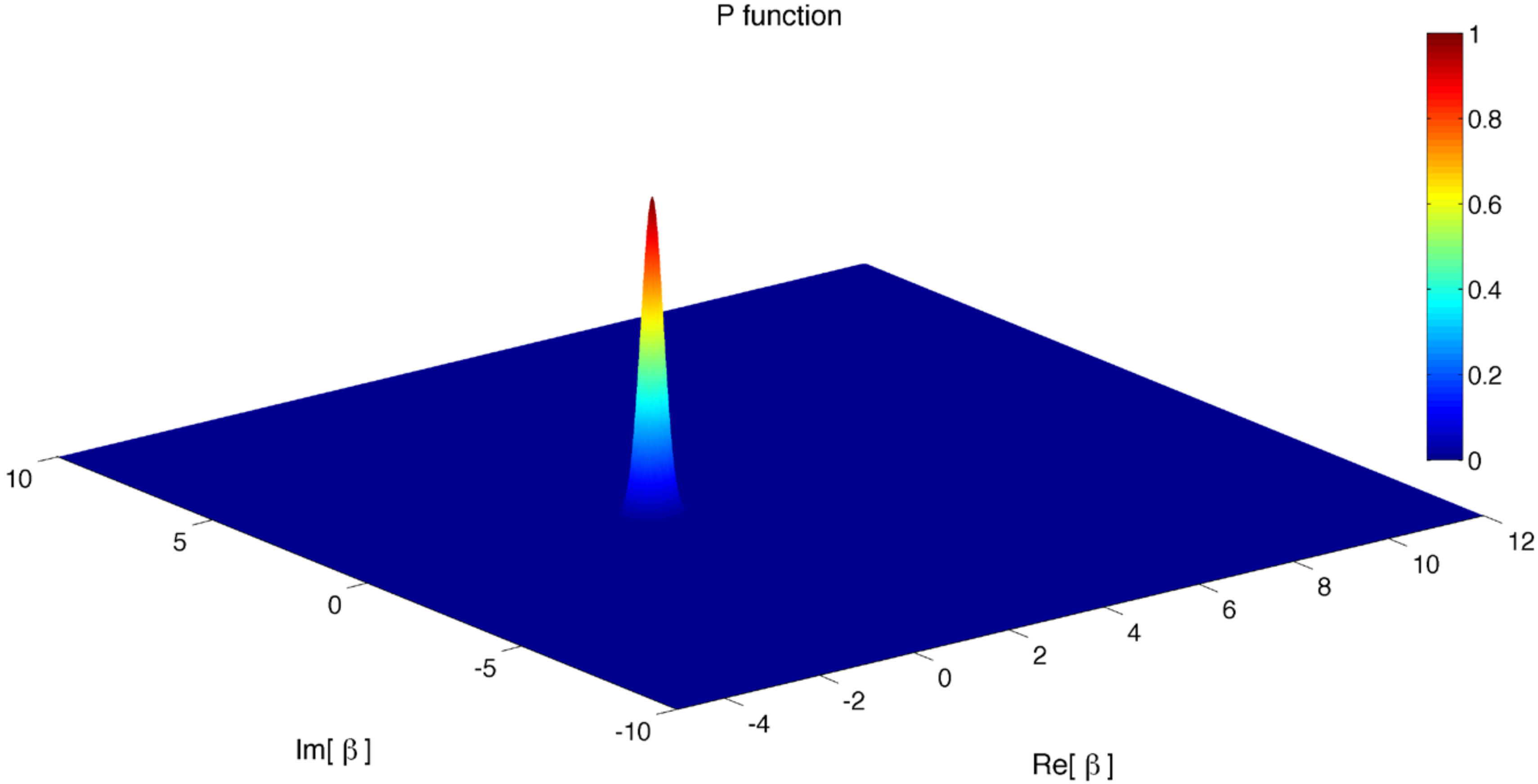
What about nonGaussian added noise?
What about higher moments of added noise?

THE PROBLEM

What are the quantum limits on the entire distribution of added noise?

Initial coherent state

Input coherent-state amplitude $\alpha = 1$



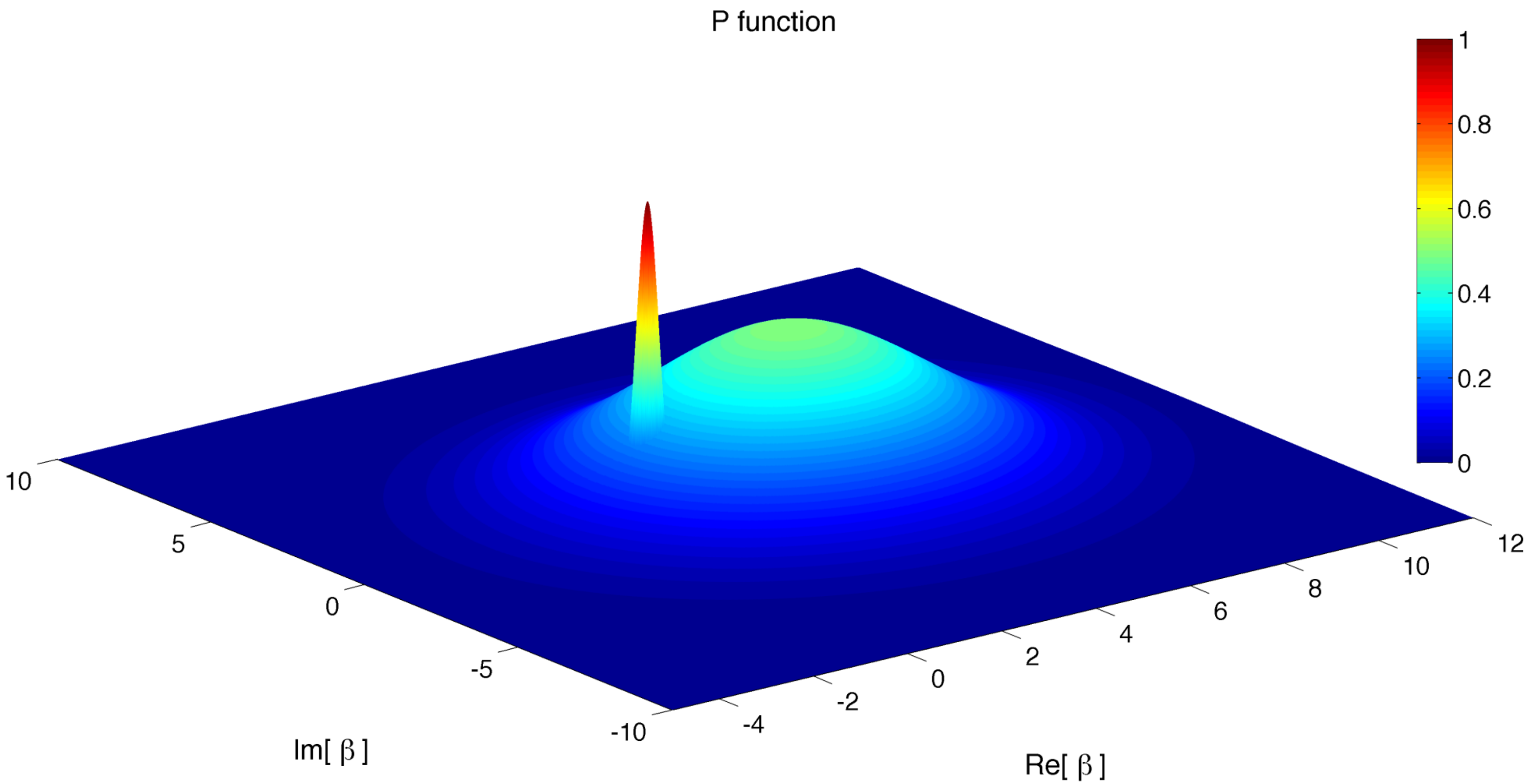
Ideal amplification of initial coherent state

Input coherent-state amplitude $\alpha = 1$

Amplitude gain $g = 4$

Output P function: Gaussian added noise with

$$\frac{\langle |\Delta L|^2 \rangle}{g^2 - 1} = 0.5$$



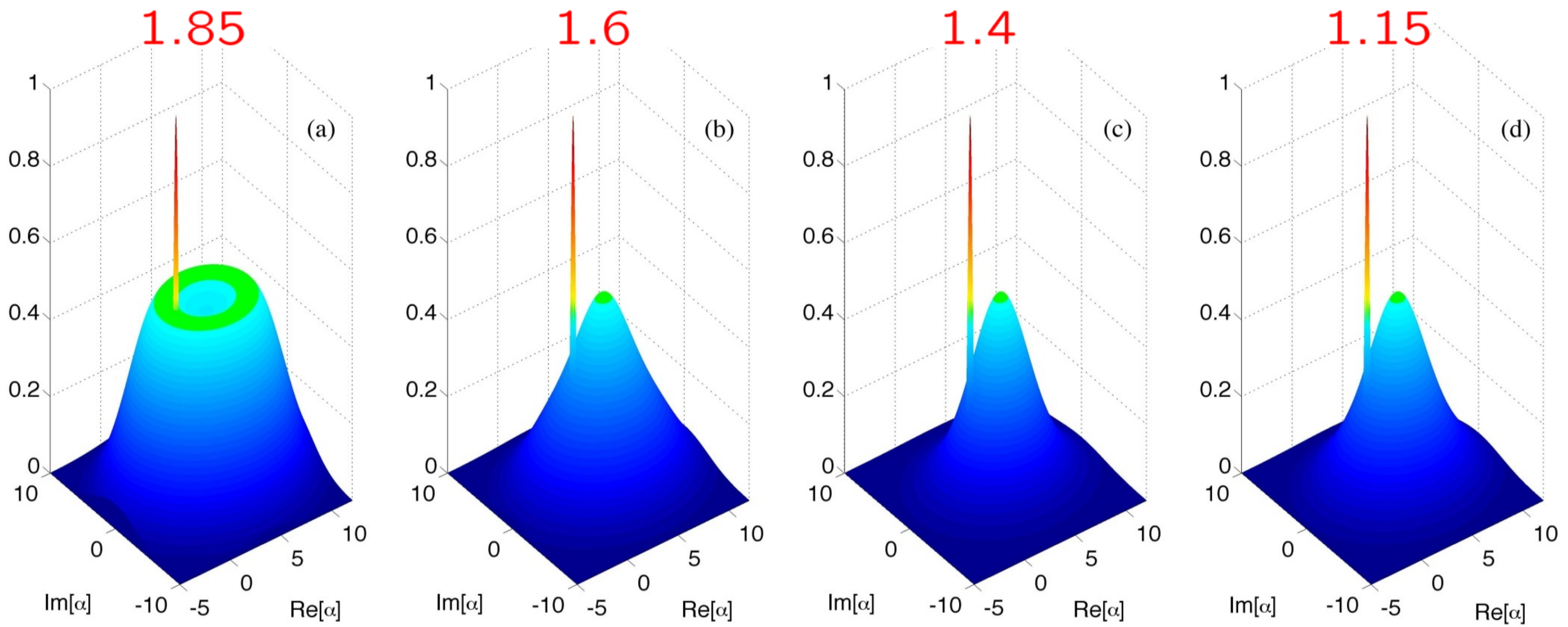
NonGaussian amplification of initial coherent state

Input coherent-state amplitude $\alpha = 1$

Amplitude gain $g = 4$

Output P functions (≥ 0):

nonGaussian added noise with $\dfrac{\langle|\Delta L|^2\rangle}{g^2 - 1} =$



Are these legitimate linear amplifiers?

II. Quantum limits on noise in phase-preserving linear amplifiers. The whole story

Oljeto Wash
Southern Utah

What is a phase-preserving linear amplifier?

$$\mathcal{A} \quad : \quad |\alpha\rangle \rightarrow |g\alpha\rangle$$

Amplification of input coherent state

$$\mathcal{B}(\rho) \quad = \quad \int d^2\beta\, \Sigma(\beta) D(a,\beta)\rho D^\dagger(a,\beta)$$

Smearing probability distribution. Smears out the amplified coherent state and includes amplified input noise and added noise.
For coherent-state input, it is the *P* function of the output.

amplifier map: $\mathcal{E} = \mathcal{B} \circ \mathcal{A}$

THE PROBLEM

Given that the amplifier map must be physical (completely positive), what are the quantum restrictions on the smearing probability distribution?

What is a phase-preserving linear amplifier?

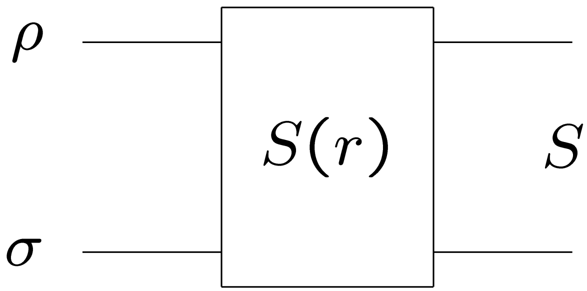
$$\begin{aligned} \mathcal{A} &: \quad |\alpha\rangle \to |g\alpha\rangle \\ \mathcal{B}(\rho) &= \int d^2\beta\, \Sigma(\beta) D(a,\beta)\rho D^\dagger(a,\beta) \\ \mathcal{E} &= \mathcal{B}\circ\mathcal{A} \end{aligned}$$

is equivalent to

$$a_{\mathrm{out}} = g a_{\mathrm{in}} + L^\dagger$$

$$[L, L^\dagger] = g^2 - 1$$

and is equivalent to



$$\mathcal{E}(\rho) = \mathrm{tr}_b\big(S(r)\rho\otimes\sigma S^\dagger(r)\big)$$

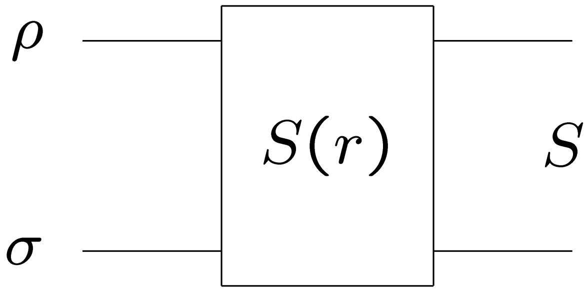
$$S(r)\rho\otimes\sigma S^\dagger(r)$$

$$\Sigma(\beta) = \frac{Q_\sigma\left(\beta/\sqrt{g^2-1}\right)}{g^2-1}$$

THE PROBLEM

Given that the amplifier map must be physical (completely positive), what are the quantum restrictions on the ancilla mode's initial "state" σ?

What is a phase-preserving linear amplifier?



$$\mathcal{E}(\rho) = \mathrm{tr}_b\big(S(r)\rho \otimes \sigma S^\dagger(r)\big)$$

$$S(r)\rho \otimes \sigma S^\dagger(r)$$

$$\Sigma(\beta) = \frac{Q_\sigma\left(\beta/\sqrt{g^2-1}\right)}{g^2-1}$$

THE ANSWER

Any phase-preserving linear amplifier is equivalent to a two-mode squeezing paramp with the smearing function being a rescaled *Q* function of a *physical* initial state σ of the auxiliary mode.

What is a phase-preserving linear amplifier?

Input coherent-state amplitude $\alpha = 1$

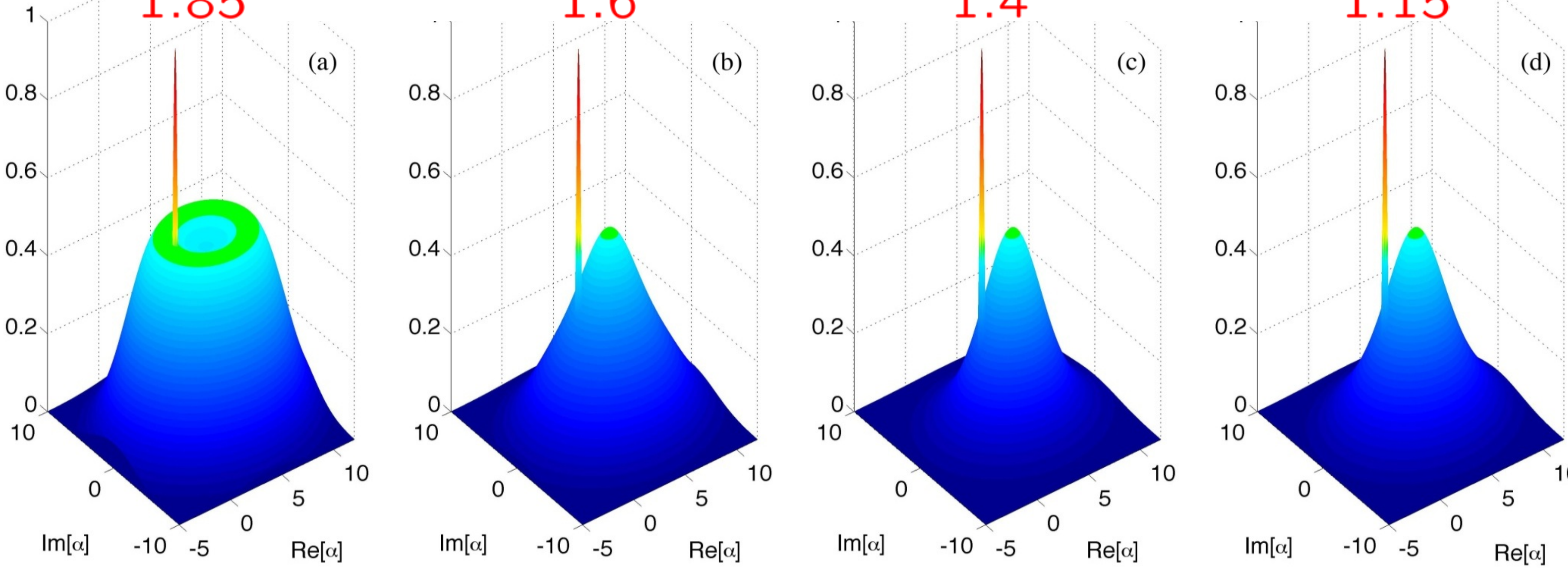
Amplitude gain $g = 4$

Output P functions (≥ 0):

nonGaussian added noise with $\dfrac{\langle|\Delta L|^2\rangle}{g^2 - 1} =$

To IV

1.85 1.6 1.4 1.15

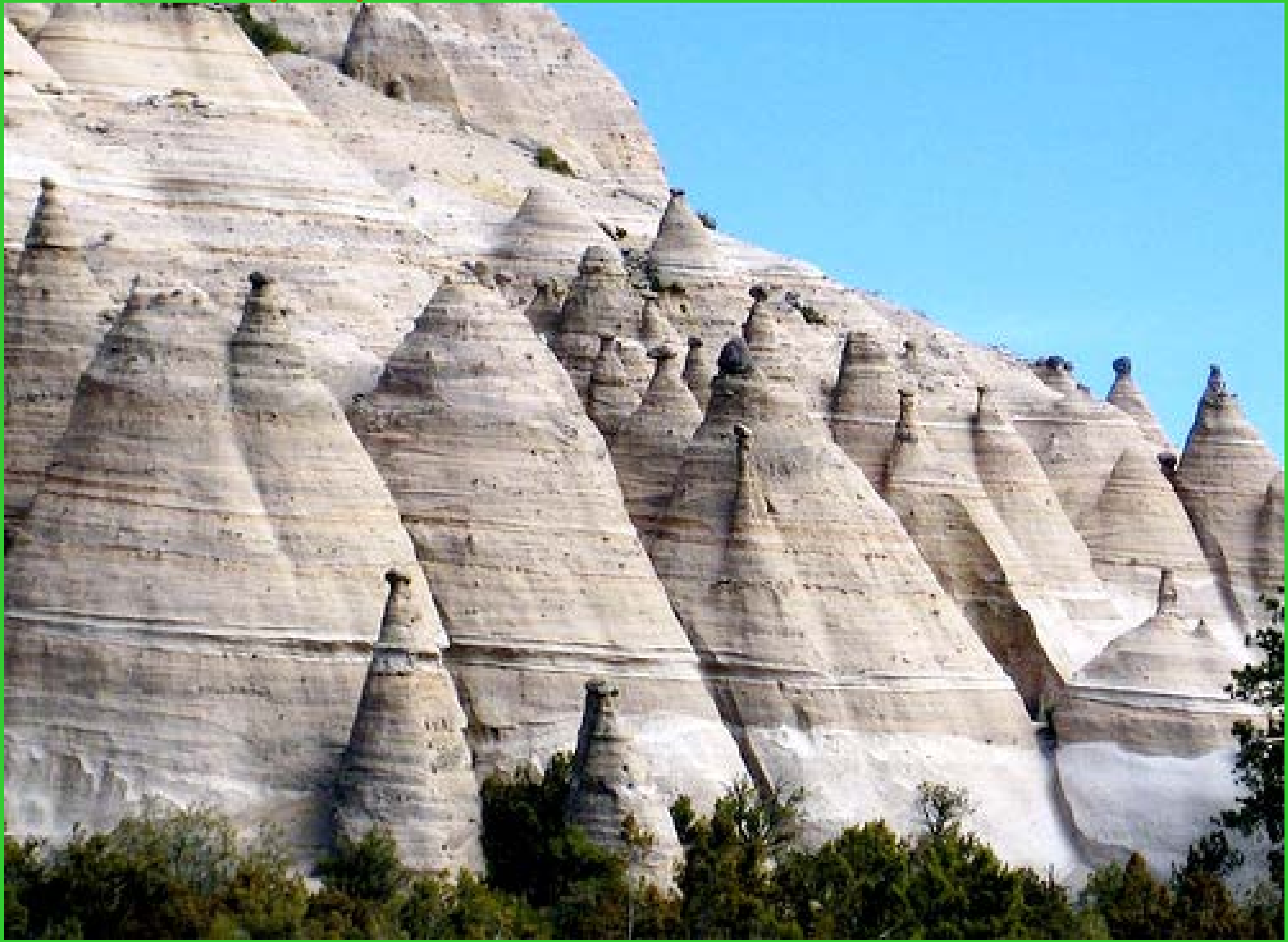


$$\sigma = (\tfrac{1}{2} - \lambda)|0\rangle\langle 0| + \lambda|1\rangle\langle 1| + \tfrac{1}{2}|2\rangle\langle 2|$$

$\lambda = 0.35$ 0.1 -0.1 -0.35

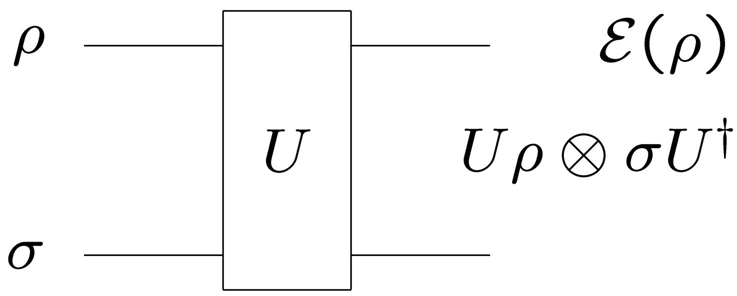
Legit Not legit

III.Completely positive maps and physical ancilla states



Tent Rocks
Kasha-Katuwe National Monument
Northern New Mexico

When does the ancilla state have to be physical?



$$\mathcal{E}(\rho) = \mathrm{tr}_B\big(U\rho\otimes\sigma U^\dagger\big)$$

THE PROBLEM

What are the restrictions on U such that $\mathcal{E}$ being completely positive implies σ is a physical density operator?

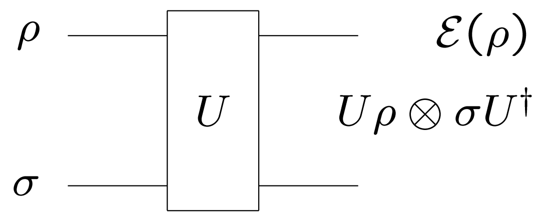
$$\mathcal{O} = \{\, O \mid \mathrm{tr}_A(U^\dagger O) = 0 \,\}$$

$$\mathcal{B} = \begin{pmatrix}\text{orthocomplement}\\ \text{of } \mathcal{O}\end{pmatrix} = \begin{pmatrix}\text{span of Schmidt}\\ \text{operators of } U\end{pmatrix}$$

$$\mathcal{C} = \{\, C = B^\dagger B \mid B \in \mathcal{B},\ \mathrm{tr}_B(C) = 1 \,\} = \begin{pmatrix}\text{subset of density}\\ \text{operators on } B\end{pmatrix}$$

Co-workers: Z. Jiang, M. Piani

When does the ancilla state have to be physical?



$$\mathcal{E}(\rho) = \mathrm{tr}_B\big(U\rho \otimes \sigma U^\dagger\big)$$

THE PROBLEM

What are the restrictions on U such that $\mathcal{E}$ being completely positive implies σ is a physical density operator?

THE ANSWER

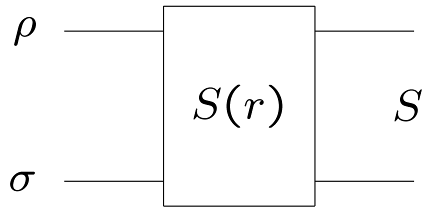
$\mathcal{C}$ contains all pure states. If $\mathcal{C}$ does not contain all pure states, any pure state not in $\mathcal{C}$ can be used as an eigenvector of σ with negative eigenvalue.

$$\mathcal{O} = \{\, O \mid \mathrm{tr}_A(U^\dagger O) = 0 \,\}$$

$$\mathcal{B} = \begin{pmatrix}\text{orthocomplement}\\ \text{of } \mathcal{O}\end{pmatrix} = \begin{pmatrix}\text{span of Schmidt}\\ \text{operators of } U\end{pmatrix}$$

$$\mathcal{C} = \{\, C = B^\dagger B \mid B \in \mathcal{B},\ \mathrm{tr}_B(C) = 1 \,\} = \begin{pmatrix}\text{subset of density}\\ \text{operators on } B\end{pmatrix}$$

Why does the ancilla state for a linear amplifier have to be physical?



$$\mathcal{E}(\rho) = \mathrm{tr}_B\big(S(r)\rho\otimes\sigma S^\dagger(r)\big)$$

$$S(r)\rho\otimes\sigma S^\dagger(r)$$

THE PROBLEM

If $\mathcal{E}$ is completely positive, does σ have to be a physical density operator?

THE ANSWER

Yes, because $\mathcal{C}$ contains all pure states.

$$\mathrm{tr}_A(S^\dagger O) = 0 \quad\Longrightarrow\quad O = 0$$

$$\mathcal{O} = \{\, O \mid \mathrm{tr}_A(S^\dagger O) = 0 \,\} = (\text{the trivial subspace})$$

$$\mathcal{B} = \begin{pmatrix}\text{orthocomplement}\\ \text{of } \mathcal{O}\end{pmatrix} = \begin{pmatrix}\text{space of all}\\ \text{operators on } B\end{pmatrix}$$

$$\mathcal{C} = \{\, C = B^\dagger B \mid B\in\mathcal{B},\ \mathrm{tr}_B(C) = 1 \,\} = \begin{pmatrix}\text{space of all density}\\ \text{operators on } B\end{pmatrix}$$

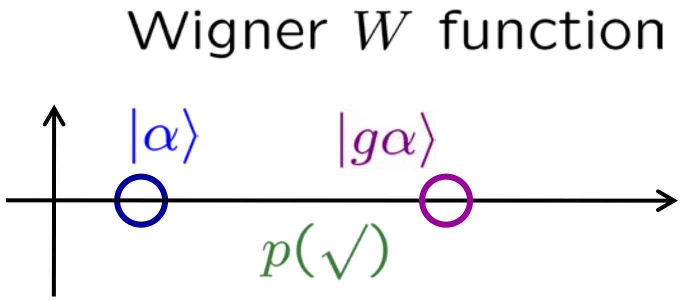
To End

IV. Nondeterministic linear amplifiers

On top of Sheepshead Peak, Truchas Peak in background
Sangre de Cristo Range
Northern New Mexico

Nondeterministic linear amplifier

Original idea (Lund and Ralph): When presented with an input coherent state, a nondeterministic linear amplifier amplifies with probability *p* and punts with probability *1 – p.*



If the probability of working is independent of input and the amplifier is described by the phase-preserving linear-amplifier map when it does work, then it is a standard linear amplifier, with the standard amount of noise, that doesn't work part of the time.

Nondeterministic linear amplifier

phase-preserving

Kraus operator $P_N K(a^\dagger a)$

Projector onto subspace of first $N+1$ number states

$$|\alpha\rangle \longrightarrow \frac{P_N K(a^\dagger a)|\alpha\rangle}{\sqrt{p(\surd|\alpha)}}$$

success probability $$p(\surd|\alpha) = \langle\alpha|K^\dagger P_N K|\alpha\rangle$$

fidelity $$F(\alpha) = \frac{|\langle g\alpha|P_N K|\alpha\rangle|^2}{p(\surd|\alpha)}$$

Nondeterministic linear amplifier

Maximize F_α given $p(\surd|\alpha)$:

$$P_N K(a^\dagger a)$$

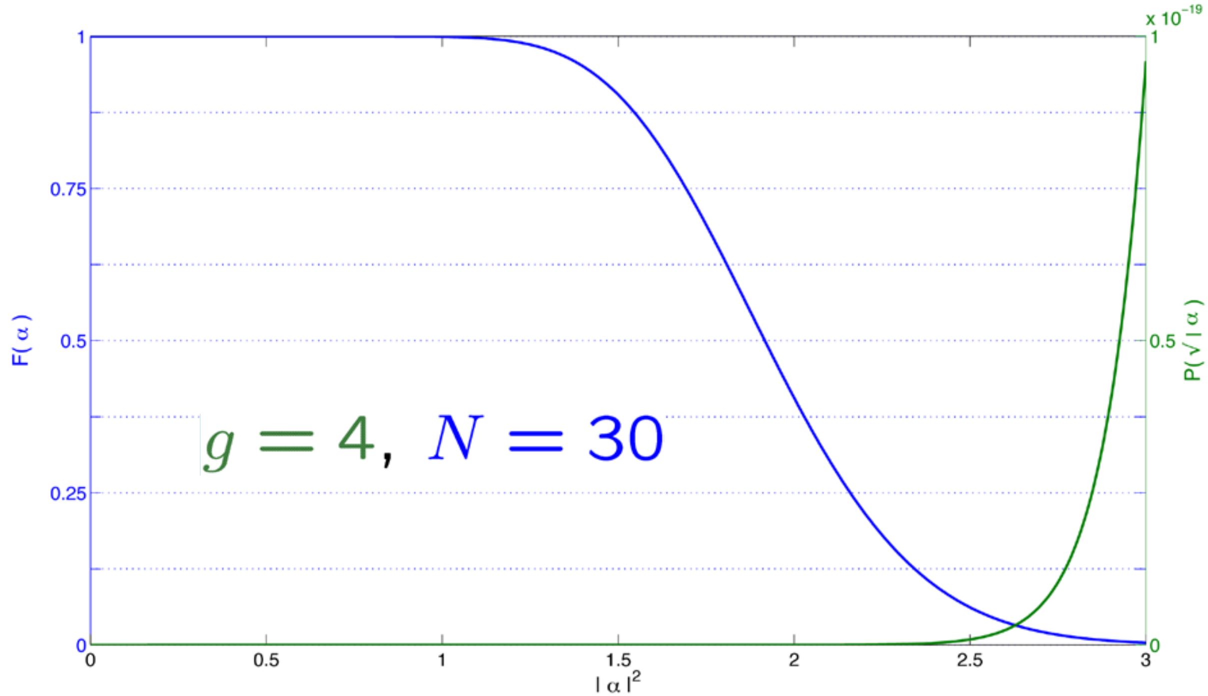
$$K = \frac{g^{a^\dagger a}}{g^N}$$

$$|\alpha\rangle \rightarrow \frac{P_N K(a^\dagger a)|\alpha\rangle}{\sqrt{p(\surd|\alpha)}} = \frac{P_N|g\alpha\rangle}{\sqrt{\langle g\alpha|P_N|g\alpha\rangle}}$$

success probability $$p(\surd|\alpha) = \langle\alpha|K^\dagger P_N K|\alpha\rangle = \frac{e^{(g^2-1)|\alpha|^2}}{g^{2N}}\langle g\alpha|P_N|g\alpha\rangle$$

fidelity $$F(\alpha) = \frac{|\langle g\alpha|P_N K|\alpha\rangle|^2}{p(\surd|\alpha)} = \langle g\alpha|P_N|g\alpha\rangle$$

Nondeterministic linear amplifier



$$\frac{g^2|\alpha|^2}{N} \ll 1: \quad F(\alpha) \simeq 1 - \frac{|g\alpha|^{2(N+1)} e^{-g^2|\alpha|^2}}{(N+1)!}$$

$$p(\surd|\alpha) \simeq \left(\frac{e^{g^2|\alpha|^2/2N}}{g}\right)^{2N}$$

$$\alpha = 0: \quad F(\alpha) = 1, \quad p(\surd|\alpha) = \frac{1}{g^{2N}}$$

That's it, folks! Thanks for your attention.

Echidna Gorge
Bungle Bungle Range
Western Australia