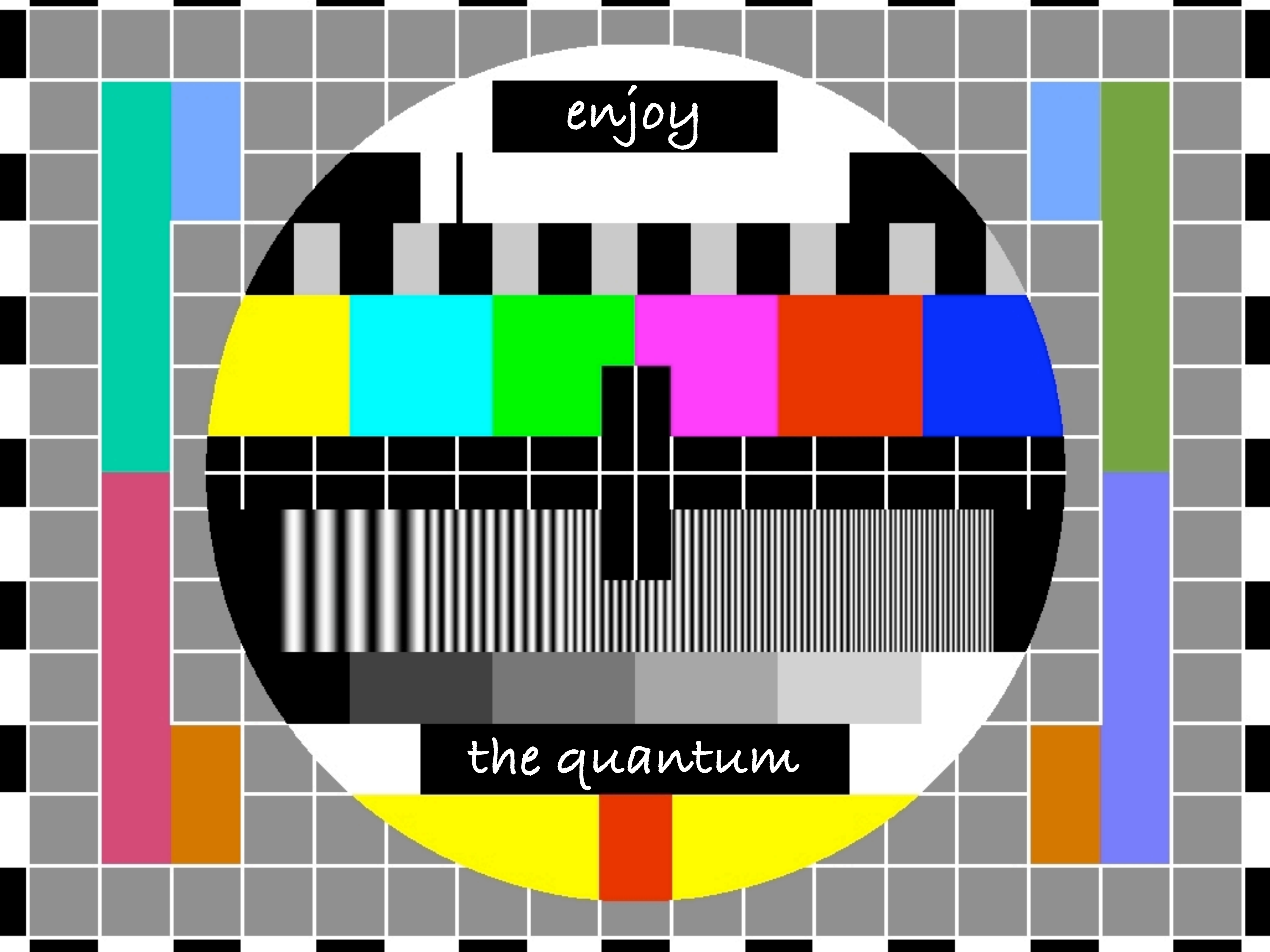
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Quantum estimation for quantum technology

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Quantum Future Technologies Conference 1.0
NASA Ames research center, 17-21 January 2012

Quantum technology ("Schroedinger's machines")

- Quantum information
  (communication and computing)

- Quantum metrology
  (calibration, interferometry, nanopositioning)

- Quantum imaging
  (ghost imaging and diffraction, quantum litography)

Quantum characterization for quantum technology

- It is highly desirable to have theoretical and experimental tools for the precise characterization of signal and devices at the quantum level

Quantum estimation

- The "resources" involved in quantum-enahnced metrology/technology are entanglement, nonlocality, entropy, interferometric phase-shift, etc..

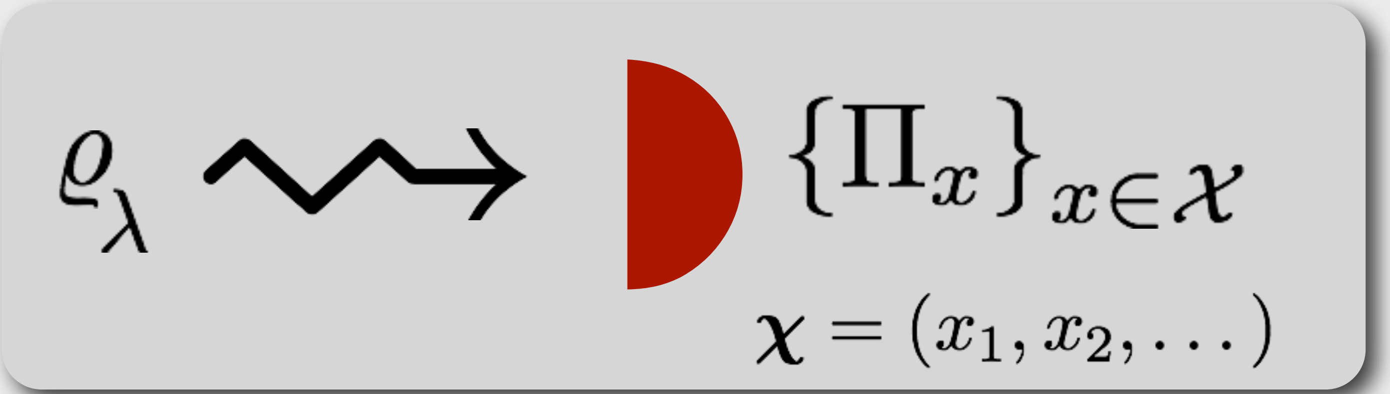
  In general they are not observable quantities in strict sense (do not correspond to a selfadjoint operator)

  - No correspondence principle
  - No uncertainty relations

Quantum estimation

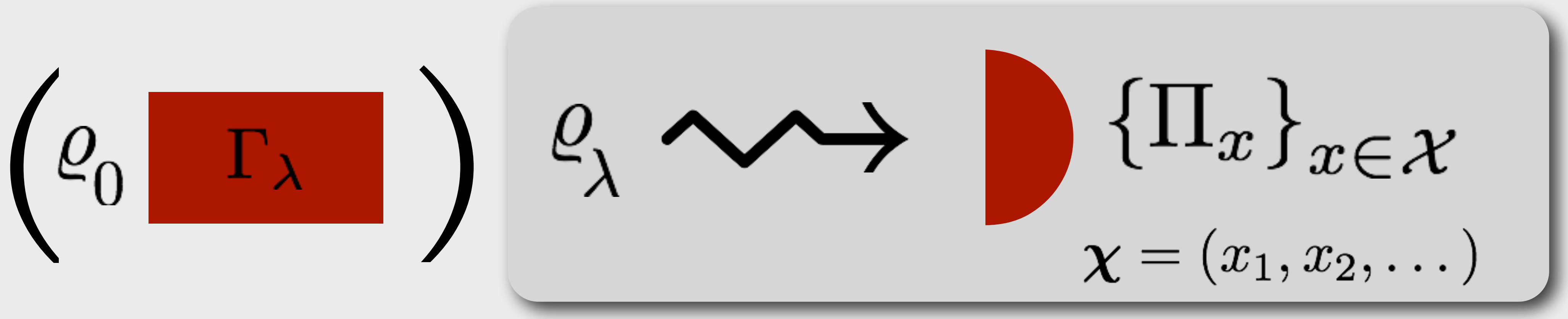
- The "resources" involved in quantum-enahnced metrology/technology are entanglement, nonlocality, entropy, interferometric phase-shift, etc..

Quantum estimation



- Optimal measurement
- Ultimate bound to precision

Quantum estimation



- Optimal measurement
- Ultimate bound to precision

Measurement and estimation

- ~~direct measurements~~
- indirect measurements

↪ influence on a different quantity

$$\boldsymbol{x} = (x_1, x_2, \dots)$$

$$\boldsymbol{x} \mapsto \widehat{\lambda} = f(\boldsymbol{x})$$

- choice of the measurement
- choice of the estimator

Cramer - Rao bound (unbiased estimators)

- variance of unbiased estimators

$$\mathrm{Var}_\lambda[\widehat{\lambda}] \geq \frac{1}{MF(\lambda)}$$

- M -> number of measurements
- F -> Fisher Information

$$F(\lambda) = \int dx\, p(x|\lambda) \left[\partial_\lambda \log p(x|\lambda)\right]^2$$

- Optimal measurement -> maximum Fisher
- Optimal estimator -> saturation of CR inequality

(Asymptotically) optimal estimators

- Bayes estimator from a posteriori distribution

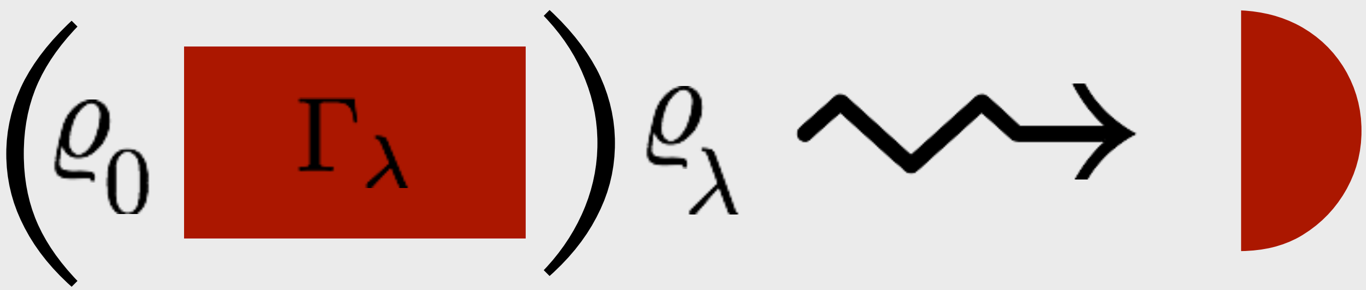
$$p(x|\lambda)p(\lambda) = p(\lambda|x)p(x)$$

Laplace Von Mises Th. $p(\lambda|\{x\}) \xrightarrow{M\gg 1} G(\lambda^*, \sigma^2)$

- MaxLik estimator(s) from the measurement likelihood

$$\mathcal{L}(x_1, x_2, ...x_M|\lambda) = \Pi_{k=1}^{M} p(x_k|\lambda)$$

Let's go quantum (1)



$$\{\Pi_x\}_{x\in\mathcal{X}}$$

$$\chi = (x_1, x_2, \dots)$$

- probability density $p(x|\lambda) = \mathrm{Tr}\left[\varrho_\lambda \, \Pi_x\right]$

- symm. log. derivative (SLD) $\dfrac{L_\lambda \varrho_\lambda + \varrho_\lambda L_\lambda}{2} = \dfrac{\partial \varrho_\lambda}{\partial \lambda}$

  selfadjoint, zero mean $\mathrm{Tr}\left[\varrho_\lambda \, L_\lambda\right] = 0$

- Fisher Information $F(\lambda) = \displaystyle\int dx \, \frac{\mathrm{Re}\left(\mathrm{Tr}\left[\varrho_\lambda \Pi_x L_\lambda\right]\right)^2}{\mathrm{Tr}[\varrho_\lambda \Pi_x]}$

Let's go quantum (2)

$$F(\lambda) \le \int dx \left| \frac{\mathrm{Tr}\,[\varrho_\lambda \Pi_x L_\lambda]}{\sqrt{\mathrm{Tr}[\varrho_\lambda \Pi_x]}} \right|^2$$

$$= \int dx \left| \mathrm{Tr} \left[\frac{\sqrt{\varrho_\lambda}\sqrt{\Pi_x}}{\sqrt{\mathrm{Tr}\,[\varrho_\lambda \Pi_x]}} \sqrt{\Pi_x} L_\lambda \sqrt{\varrho_\lambda} \right] \right|^2$$

$$\le \int dx \, \mathrm{Tr}\,[\Pi_x L_\lambda \varrho_\lambda L_\lambda]$$

$$= \mathrm{Tr}[L_\lambda \varrho_\lambda L_\lambda] = \mathrm{Tr}[\varrho_\lambda L_\lambda^2]$$

(Braunstein and Caves 1994)

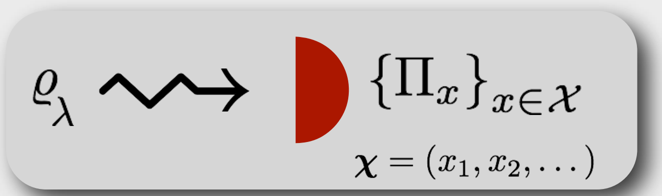
- Fisher vs Quantum Fisher

$$F(\lambda) \le H(\lambda) \equiv \mathrm{Tr}[\varrho_\lambda L_\lambda^2] = \mathrm{Tr}[\partial_\lambda \varrho_\lambda L_\lambda]$$

- Quantum Cramer-Rao bound $\mathrm{Var}(\lambda) \ge \frac{1}{M H(\lambda)}$

Optimal quantum measurement (1)

- the optimal measurement is a projective one, the spectral measure is built with the eigenstates of the SLD

- 

  optimal quantum measurement:
  SLD + classical postprocessing (Bayesian, ML)

General formulas (basis indepedent)

$$\frac{L_\lambda \varrho_\lambda + \varrho_\lambda L_\lambda}{2} = \frac{\partial \varrho_\lambda}{\partial \lambda}$$ Lyapunov equation

- Symmetric logarithmic derivative

$$L_\lambda = 2 \int_0^\infty dt\, \exp\{-\varrho_\lambda t\}\, \partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}$$

- Quantum Fisher Information

$$H(\lambda) = 2 \int_0^\infty dt\, \mathrm{Tr}\left[\partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}\, \partial_\lambda \varrho_\lambda \exp\{-\varrho_\lambda t\}\right]$$

General formulas

- Family of quantum states

$$\varrho_\lambda = \sum_n \varrho_n |\psi_n\rangle\langle\psi_n|$$

- Symmetric logarithmic derivative

$$L_\lambda = \sum_p \frac{\partial_\lambda \varrho_p}{\varrho_p} |\psi_p\rangle\langle\psi_p| + 2\sum_{n\neq m} \frac{\varrho_n - \varrho_m}{\varrho_n + \varrho_m} \langle\psi_m|\partial_\lambda\psi_n\rangle\, |\psi_m\rangle\langle\psi_n|$$

- Quantum Fisher Information

$$H(\lambda) = \sum_p \frac{(\partial_\lambda \varrho_p)^2}{\varrho_p} + 2\sum_{n\neq m} \frac{(\varrho_n - \varrho_m)^2}{\varrho_n + \varrho_m} |\langle\psi_m|\partial_\lambda\psi_n\rangle|^2$$

Optimal quantum measurement (2)

ultimate bound on precision $\mathrm{Var}(\lambda) \geq \frac{1}{MH(\lambda)}$

(local quantum estimation theory)

- feedback assisted/adaptive measurements

- one-step adaptive procedure: rough estimate of the parameter on a small fraction of copies + measurement of SLD on the rest of the copies

unitary families of quantum states

$$\varrho_\lambda = U_\lambda\, \varrho_0\, U_\lambda^\dagger \qquad \varrho_0 = \sum \varrho_n |\varphi_n\rangle\langle\varphi_n|$$

$$U_\lambda = \exp\{-i\lambda G\} \qquad \partial_\lambda \varrho_\lambda = U_\lambda [G, \varrho_0] U_\lambda^\dagger$$

- covariance of SLD $\quad L_\lambda = U_\lambda\, L_0\, U_\lambda^\dagger$

$$L_0 = 2 \sum_{n,m} \frac{\langle\varphi_m|[G, \varrho_0]|\varphi_n\rangle}{\varrho_n + \varrho_m} |\varphi_n\rangle\langle\varphi_m|$$

- QFI is independent on the value of the parameter

$$H = \mathrm{Tr}\left[\varrho_0\, L_0^2\right] = 2 \sum_{n\neq m} \frac{(\varrho_n - \varrho_m)^2}{\varrho_n + \varrho_m} \langle\varphi_m|G|\varphi_n\rangle^2$$

parameter-based uncertainty relations

- pure states $H = 4\langle\psi_0|\Delta G^2|\psi_0\rangle$

$$\mathrm{Var}(\lambda)\langle\Delta G^2\rangle \geq \frac{1}{4M}$$

parameter-based uncertainty relations

- pure states $H = 4\langle\psi_0|\Delta G^2|\psi_0\rangle$

$$\mathrm{Var}(\lambda)\langle\Delta G^2\rangle \geq \frac{1}{4M}$$

- mixed states

$$H = 4\,\mathrm{Tr}\left[\Delta G^2 \varrho_0\right] + 4\sum_n \varrho_n \langle\varphi_n|\langle G\rangle^2 - 2GK^{(n)}G|\varphi_n\rangle$$

$$K^{(n)} = \sum_m \frac{\varrho_m}{\varrho_n + \varrho_m}|\varphi_m\rangle\langle\varphi_m| \xrightarrow{\varrho_0 \to |\varphi_0\rangle\langle\varphi_0|} \frac{1}{2}|\varphi_0\rangle\langle\varphi_0|$$

$$\mathrm{Var}(\lambda)\langle\Delta G^2\rangle \geq \frac{1}{4M}\left[1 + \sum_n \varrho_n \langle\varphi_n|\langle G\rangle^2 - 2GK^{(n)}G|\varphi_n\rangle\right]^{-1}$$

estimability of a parameter

- signal-to-noise ratio (single measurement)

$$R_\lambda = \frac{\lambda^2}{\mathrm{Var}(\lambda)} \leq Q_\lambda \equiv \lambda^2 H(\lambda)$$

- relative error for a 3σ confidence interval (after M measurements)

$$\delta^2 = \frac{9\mathrm{Var}(\lambda)}{M\lambda^2} = \frac{9}{M}\frac{1}{Q_\lambda} = \frac{9}{M\lambda^2 H(\lambda)}$$

- # of meas to achieve a given relative error

$$M_\delta = \frac{9}{\delta^2}\frac{1}{Q_\lambda}$$

estimability of a parameter: the unitary case

$$\varrho_\lambda = U_\lambda\, \varrho_0\, U_\lambda^\dagger$$

- QFI is independent on the value of the parameter
- (Any) estimation procedure cannot be efficient for small value of the parameter

$$Q_\lambda \propto \lambda^2 \qquad\qquad M_\delta \propto \frac{1}{\delta^2 \lambda^2}$$

A nonunitary example: estimation of loss

- Master equation $\frac{d\varrho_\phi}{d\phi} = \tan\phi \, L[a]\varrho_\phi \quad \exp\{-\gamma t\} = \cos^2\phi$

$$L[a]\varrho = 2a^\dagger \varrho a - a^\dagger a \varrho - \varrho a^\dagger a$$

- absorption
- propagation in a noisy channel (T=0)

A nonunitary example: estimation of loss

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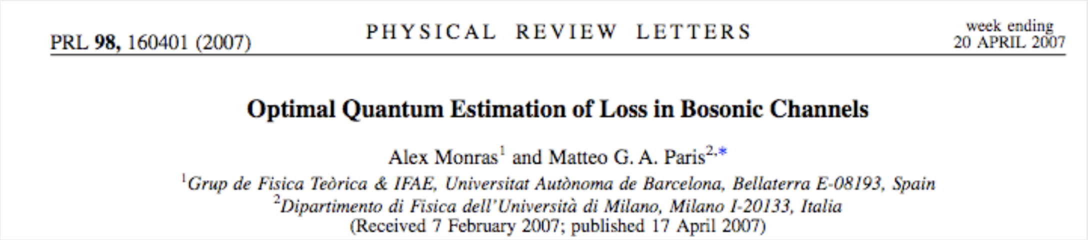
$$L[a]\varrho = 2a^\dagger \varrho a - a^\dagger a \varrho - \varrho a^\dagger a$$

- absorption
- propagation in a noisy channel (T=0)

- optimal measurement: Gaussian operations + photon count.
- ultimate precision $\mathrm{Var}_\gamma[\hat{\gamma}] \to \frac{\gamma}{\bar{n} M t} + \mathrm{O}(\gamma^2)$

proportional to the loss parameter itself !


PRL **98,** 160401 (2007) PHYSICAL REVIEW LETTERS week ending 20 APRIL 2007

Optimal Quantum Estimation of Loss in Bosonic Channels

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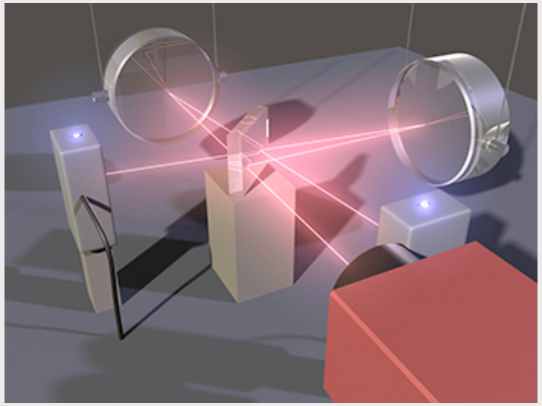
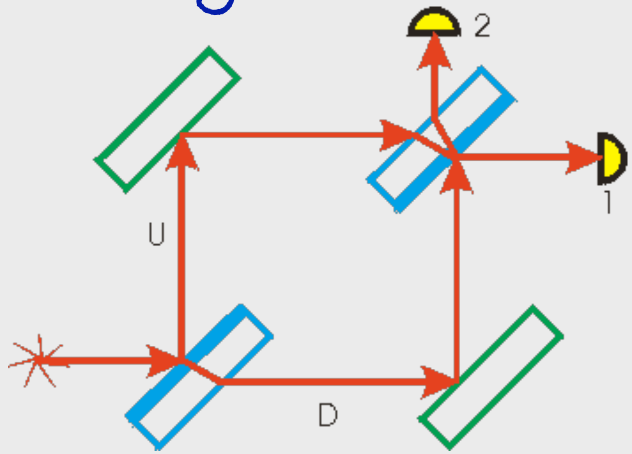
Quantum Interferometry

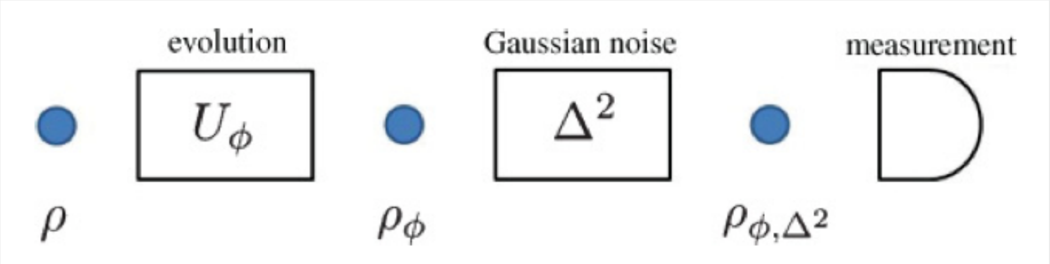




use of quantum states of light to improve sensitivity

- Optimization over input states (Caves 1981)
- Effects of detection noise
- Effects of losses
- Multiple interference
- Fixed number of particles, atomic interf

Estimation of phase in the presence of phase diffusion



$$U_\phi = \exp\{-i\, a^\dagger a\, \phi\}$$

$$\varrho_\phi = \mathcal{N}_\Delta[U_\phi\, \varrho\, U_\phi^\dagger] = U_\phi\, \mathcal{N}_\Delta[\varrho]U_\phi^\dagger$$

$$\mathcal{N}_\Delta[\varrho] = \sum_{nm} e^{-\Delta^2(n-m)^2}\, \varrho_{nm}\, |n\rangle\langle m|$$

- In the noiseless case the optimal probe is the squeezed vacuum and $H = 8(N^2 + N)$ (Monras 2006)

Estimation of phase in the presence of phase diffusion

- In the presence of noise we have (approximate) scaling laws

$$H(N,\Delta) \simeq k^2 H(N/k, k\Delta) \qquad \beta_{\rm opt}(N,\Delta) \simeq \beta_{\rm opt}(N/k, k\Delta)$$

- Homodyning is nearly optimal
  for low and high noise

PRL 106, 153603 (2011) PHYSICAL REVIEW LETTERS week ending 15 APRIL 2011

Optical Phase Estimation in the Presence of Phase Diffusion

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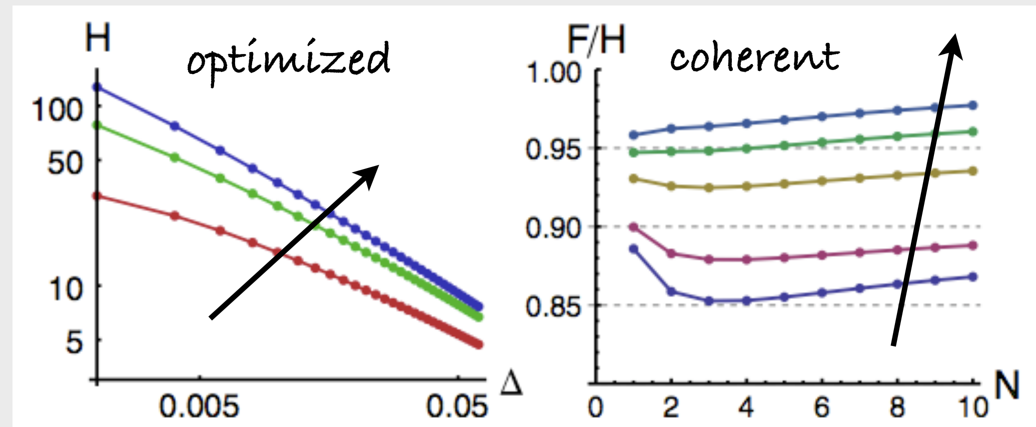

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PRL 106, 153603 (2011) PHYSICAL REVIEW LETTERS week ending 15 APRIL 2011

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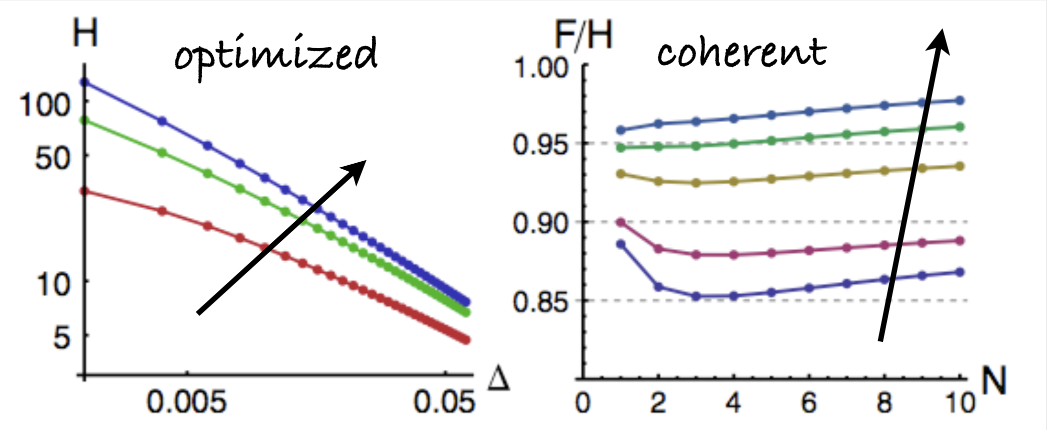
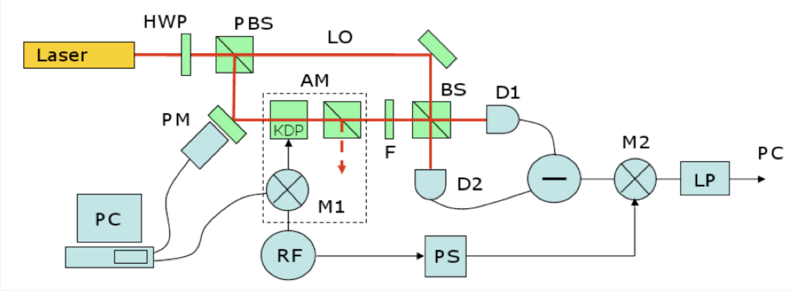
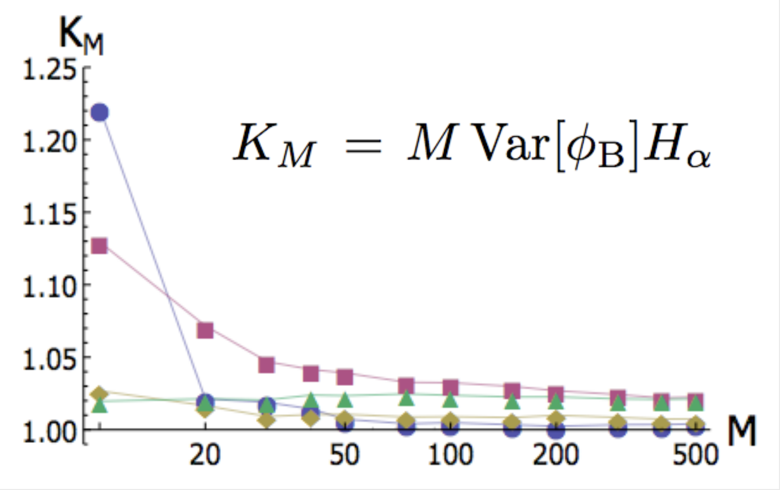


- Experiments with coherent states (large noise)





Bayes estim. is optimal for small samples

Estimation of entanglement

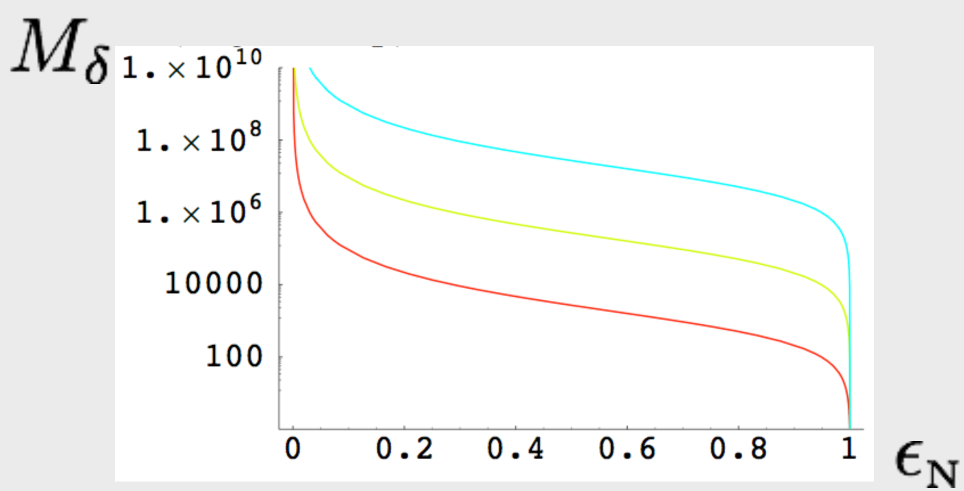
- pure states (Schmidt decomposition)

$$|\Psi_q\rangle = \sqrt{q}|0\rangle_A|0\rangle_B + \sqrt{1-q}|1\rangle_A|1\rangle_B$$

- entanglement measure: function of q (negativity, linear and VN entropy)

- QSNR is vanishing for vanishing entanglement

$$Q(\epsilon_N) = \frac{\epsilon_N^2}{1-\epsilon_N^2} \overset{\epsilon_N \to 0}{\sim} \epsilon_N^2$$



The multiparametric case

- QFI matrix $$\boldsymbol{H}(\boldsymbol{\lambda})_{\mu\nu} = \mathrm{Tr}\left[\varrho_{\boldsymbol{\lambda}} \frac{L_\mu L_\nu + L_\nu L_\mu}{2}\right]$$

- bound on covariance (not achievable) $\mathrm{Cov}[\gamma]_{ij} = \langle \lambda_i \lambda_j \rangle - \langle \lambda_i \rangle \langle \lambda_j \rangle$
$$\mathrm{Cov}[\gamma] \geq \frac{1}{M} \boldsymbol{H}(\boldsymbol{\lambda})^{-1}$$

- single parameter (achievable) $$\mathrm{Var}(\lambda_\mu) \geq \frac{1}{M} (\boldsymbol{H}^{-1})_{\mu\mu}$$

repametrization $\widetilde{\boldsymbol{\lambda}} = \{\widetilde{\lambda}_j = \widetilde{\lambda}_j(\boldsymbol{\lambda})\}$ $\widetilde{\lambda}_1 \equiv g(\boldsymbol{\lambda})$

$$\widetilde{L}_\mu = \sum_\nu B_{\mu\nu} L_\nu \qquad \widetilde{\boldsymbol{H}} = \boldsymbol{B}\boldsymbol{H}\boldsymbol{B}^T \qquad B_{\mu\nu} = \partial \lambda_\nu / \partial \widetilde{\lambda}_\mu$$

Estimation of entanglement

- different measures (negativity, entropy, distance) and families of states (qubit and CV)

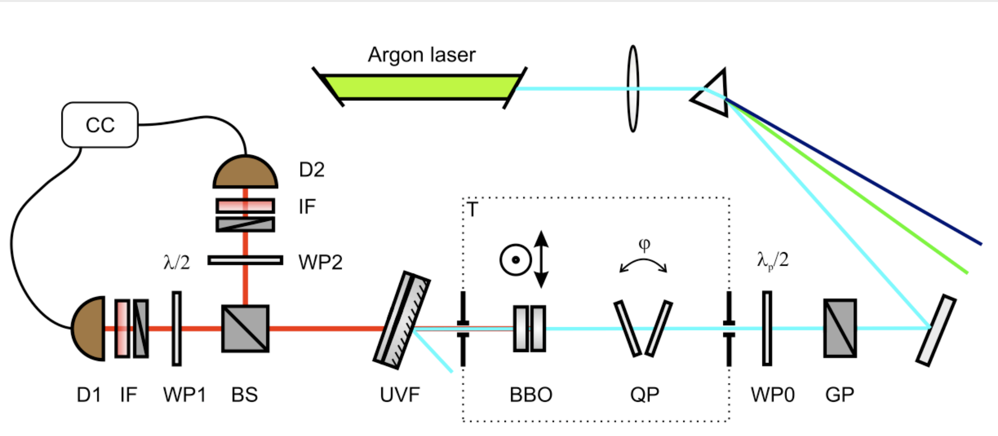
- QFI is increasing with entanglement
  QSNR diverges for maximal entanglement

- Qubit: QSNR is vanishing for vanishing entanglement
  Estimation of (low) entanglement is inherently inefficient

- CV: appropriate entanglement measure may achieve efficient estimation

M. G. Genoni, P. Giorda and M. G. A. Paris PRA 2008

Estimation of entanglement (@INRIM)



$$|\psi_\phi\rangle = \cos\phi|HH\rangle + \sin\phi|VV\rangle$$

$$D_\phi = \cos^2\phi|HH\rangle\langle HH| + \sin^2\phi|VV\rangle\langle VV|$$

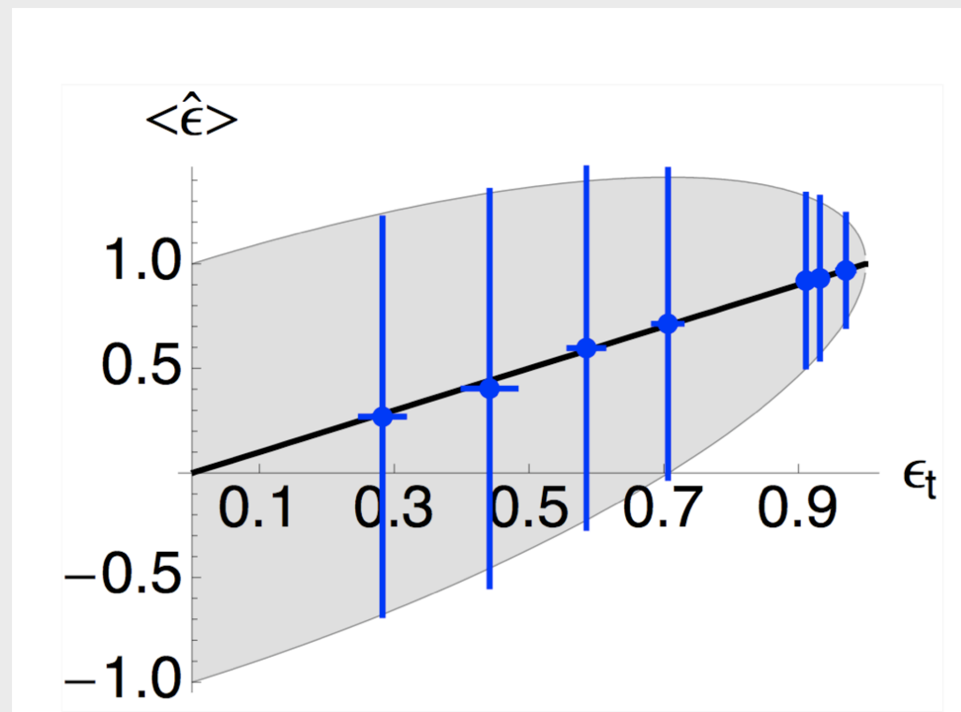
$$\varrho_\epsilon = p|\psi_\phi\rangle\langle\psi_\phi| + (1-p)D_\phi$$

$$\epsilon = p\sin 2\phi$$

optimal estimation by visibility measurements

Fisher information is monotone with entanglement

Estimation of (low) entanglement is inherently inefficient



PRL 104, 100501 (2010) PHYSICAL REVIEW LETTERS week ending 12 MARCH 2010

Experimental Estimation of Entanglement at the Quantum Limit

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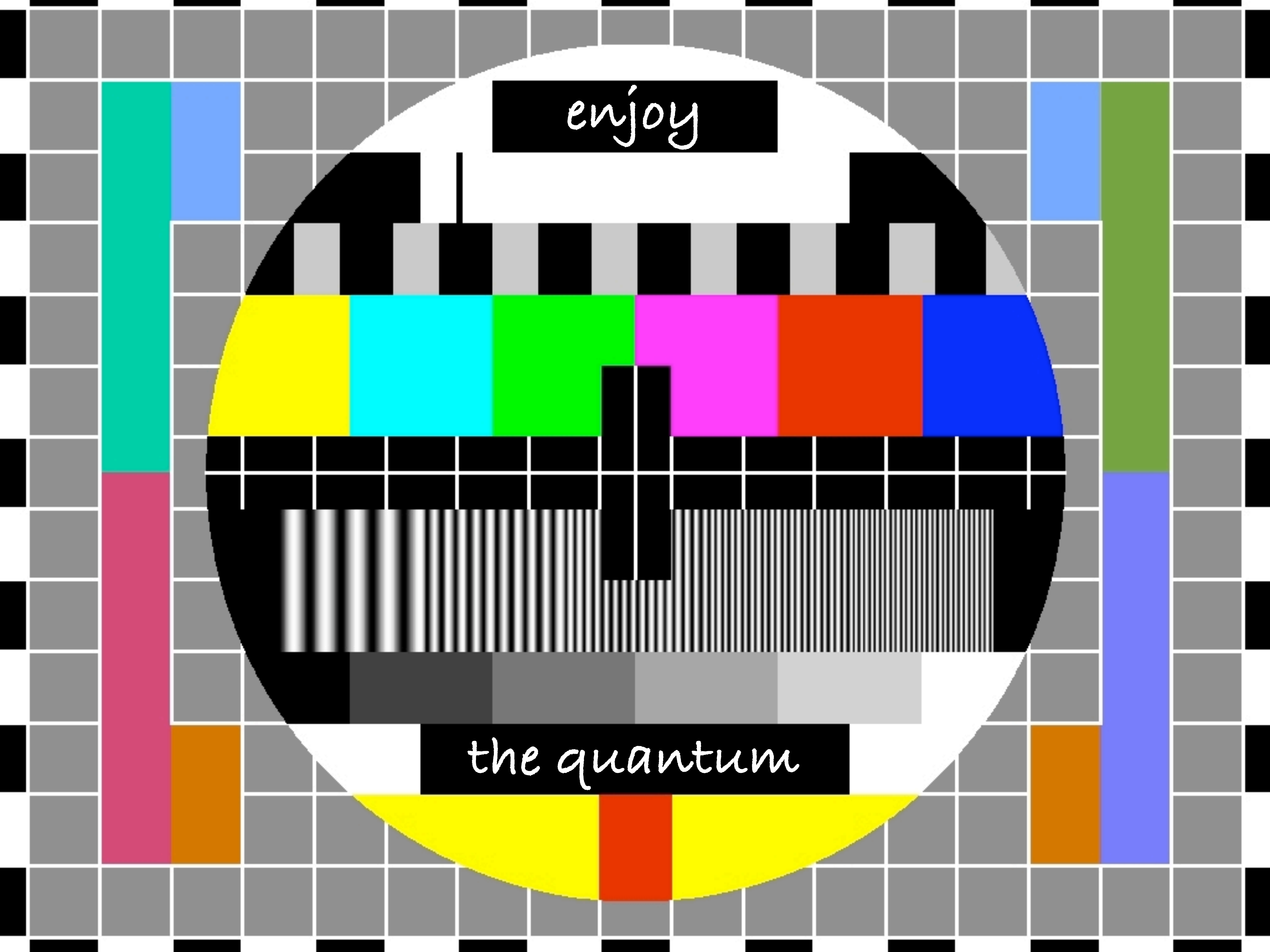
Summary

- Quantum estimation for quantum technology:
  - Optimal quantum measurement in terms of SLD and ultimate bounds to the precision of the estimation of any quantity of interest including non-observables
  - intrinsic estimability of a parameter
  - classical and quantum contributions to uncertainty
- Quantum estimation of nonobservable quantities
  - coupling constants
  - interferometry
  - entanglement
  - ...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the quantum

(classical) Bayesian estimators (1)

- Bayes theorem $$p(x|\lambda)p(\lambda) = p(\lambda|x)p(x)$$

- M indipendent events: a posteriori distribution

$$p(\lambda|\{x\}) = \frac{1}{N}\prod_{k=1}^{M} p(x_k|\lambda) \qquad N = \int d\lambda \prod_{k=1}^{M} p(x_k|\lambda)$$

- Bayesian estimator: $$\lambda_B = \int d\lambda\, \lambda\, p(\lambda|\{x\})$$

mean of the a posteriori distribution

(classical) Bayesian estimators (2)

- Laplace - Bernstein - von Mises theorem

$$p(\lambda|\{x\}) \xrightarrow{M \gg 1} G(\lambda^*, \sigma^2)$$

- Bayes estimator is asymptotically efficient

$$\sigma^2 = \frac{1}{MF(\lambda^*)}$$

MaxLik estimation

- Probability distribution $p(x|\lambda)$
- Random sample $x_1, x_2, ..., x_M$
- Joint probability of the sample

$$\mathcal{L}(x_1, x_2, ...x_M|\lambda) = \Pi_{k=1}^{M} p(x_k|\lambda)$$

Maxlik estimation ➞ take the value of the parameters which maximize the likelihood of the observed data