Transverse Beam Transfer Function for Overlapping Betatron Sidebands

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I. Introduction

At the microwave frequencies typical of most of the stochastic cooling systems in the antiproton source, the width of the betatron dipole resonances is large enough that the tails of the nearest neighbor resonance affects the beam amplitude and phase response observed during various diagnostic cooling system transfer function measurements. This effect should manifest itself throughout the entire 2-4 GHz bandwidth of the Recycler transverse stochastic cooling systems. The effects of overlapping sidebands impact cooling system phasing, signal suppression, and overall cooling effectiveness at high frequencies.

The problem of overlapping sidebands has been considered before [1]. The purpose of this note is to fill in some of the tedious mathematical detail and illustrate this effect with a numerical model relevant to the Recycler stochastic cooling systems.

II. Single Particle Response

The transverse beam transfer function is the normalized response of the beam to a transverse kick, which varies sinusoidally in time. The kick is delivered through a dipole kicker appropriate to the frequency and power being applied to the beam. The response is detected by a pickup sensitive to the dipole moment of the beam at the frequency of the kick (see Figure 1).

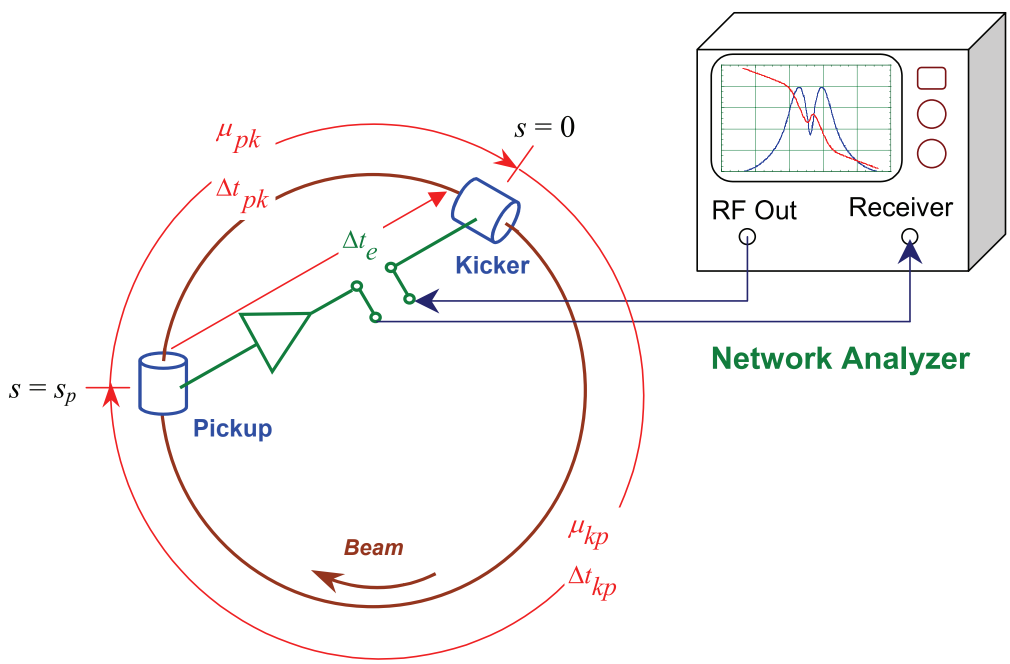

Figure 1 Setup for a beam transfer function measurement. The various phase advances and delays are shown. The total phase advance around the ring is $2\pi Q = \mu_{kp} + \mu_{pk}$. Note: the electronic delay, Δt_e, does not include the delay in the cables that connect the network analyzer to the stochastic cooling electronics.

The basic strategy used to calculate the beam response is to follow the approach used to calculate the closed orbit distortion to a single dipole error. In this case the dipole error, $\theta(t)$, varies sinusoidally in time. $\theta(t)$ is then given by:

$$\theta(t) = \hat{\theta} e^{i\omega t} \tag{1}$$

ω is the angular frequency of the applied beam excitation, and $\hat{\theta}$ is the complex phasor amplitude of the excitation. The single particle equation of motion for the beam in the y plane is Hill's equation:

$$\frac{d^2 y}{ds^2} + K(s) y = 0 \tag{2}$$

s is the longitudinal coordinate measured from the kicker. $K(s)$ is the normalized gradient function. We are looking for a time dependent solution to equation (2), $y(s,t)$, satisfying the following boundary conditions:

$$\begin{aligned} y(0,t) &= y(L,t) \\ y'(0,t) &= y'(L,t) + \hat{\theta} e^{i\omega t} \end{aligned} \tag{3}$$

L is the circumference of the accelerator. The kick is introduced by way of the boundary conditions at the kicker. The change in angular divergence of the beam at the location of the kicker must be equal to the applied kick at that moment in time.

Some care must be taken in evaluating the longitudinal derivatives since s and t are related through the particle velocity, v ($s = vt$). $y'(s,t)$ is given by:

$$\frac{dy(s,t)}{ds} = \frac{\partial y(s,t)}{\partial s} + \frac{1}{v}\frac{\partial y(s,t)}{\partial t} \tag{4}$$

Using equation (4) when taking derivatives, equation (2) becomes:

$$\frac{\partial^2 y(s,t)}{\partial s^2} + \frac{2}{v}\frac{\partial^2 y(s,t)}{\partial s \partial t} + \frac{1}{v^2}\frac{\partial^2 y(s,t)}{\partial t^2} + K(s) y(s,t) = 0 \tag{5}$$

The familiar solution to the time independent problem is given by:

$$y_{\pm}(s) = \sqrt{\varepsilon \beta(s)}\, e^{\pm i\mu(s)} \tag{6}$$

ε is the transverse emittance of the particle, $\beta(s)$ is the beta-function value at the location where the particle position is being observed, and $\mu(s)$ is the betatron phase advance from the kicker to longitudinal location s. We are looking for a solution to the time dependent problem that consists of two waves (corresponding to the so called fast and slow wave dipole excitations) propagating from the kicker to the pickup. This suggests a solution of the form:

$$\begin{aligned} y(s,t) &= A_+(\omega) y_+(s) e^{i\omega\left(t - \frac{s}{v}\right)} + A_-(\omega) y_-(s) e^{i\omega\left(t - \frac{s}{v}\right)} \\ &= \sqrt{\varepsilon\beta(s)} \left\{ A_+(\omega) e^{i\left[\mu(s) + \omega\left(t - \frac{s}{v}\right)\right]} + A_-(\omega) e^{-i\left[\mu(s) - \omega\left(t - \frac{s}{v}\right)\right]} \right\} \end{aligned} \tag{7}$$

The values of $A_+(\omega)$ and $A_-(\omega)$ are determined by substitution of the solution (7) into the time dependant Hill's equation (5) and applying the boundary conditions given by equations (3). This procedure is carried out in detail in Appendix A. The single particle solution is:

$$y(s,t)=\frac{\sqrt{\beta_k\beta(s)}}{4}\hat{\theta}\left\{\frac{e^{-i\left[\mu(s)-\pi\left(\frac{\omega}{\omega_0}+Q\right)-\omega\left(t-\frac{s}{v}\right)\right]}}{\sin\pi\left(\frac{\omega}{\omega_0}+Q\right)}-\frac{e^{i\left[\mu(s)+\pi\left(\frac{\omega}{\omega_0}-Q\right)+\omega\left(t-\frac{s}{v}\right)\right]}}{\sin\pi\left(\frac{\omega}{\omega_0}-Q\right)}\right\} \tag{8}$$

ω_0 is the angular revolution frequency of the beam particle ($\omega_0 = 2\pi f_{rev}$), β_k is the beta-function value at the kicker and Q is the betatron tune in the transverse plane of interest.

The phase velocity of the first term on the right hand side of equation (8), v_s, is given by:

$$v_s=\frac{v\omega\beta(s)}{\omega\beta(s)+v} \tag{9}$$

The phase velocity of the second term, v_f, is given by:

$$v_f=\frac{v\omega\beta(s)}{\omega\beta(s)-v} \tag{10}$$

Since $v_s < v$, the first term in equation (8) is usually called the slow-wave solution. Correspondingly, since $v_f > v$, the second term is designated the fast-wave solution.

Factoring out the time dependence in (8) gives:

$$y(s,t)=\hat{y}(\omega;\omega_0)e^{i\omega t} \tag{11}$$

The phasor amplitude $\hat{y}(\omega;\omega_0)$ is given by:

$$\hat{y}(\omega;\omega_0)=\frac{\sqrt{\beta_k\beta(s)}}{4}\hat{\theta}\left\{\frac{e^{-i\left[\mu(s)-\pi\left(\frac{\omega}{\omega_0}+Q\right)+\omega\frac{s}{v}\right]}}{\sin\pi\left(\frac{\omega}{\omega_0}+Q\right)}-\frac{e^{i\left[\mu(s)+\pi\left(\frac{\omega}{\omega_0}-Q\right)-\omega\frac{s}{v}\right]}}{\sin\pi\left(\frac{\omega}{\omega_0}-Q\right)}\right\} \tag{12}$$

The beam response is detected by a transverse stochastic cooling system pickup array located longitudinally at position s_p relative to the kicker. The beam response of equation (12) is referred to the pickup as follows: First, the pickup response is delayed by the kicker to pickup beam transit time: $\Delta t_{kp}=s_p/v$. Thus, replacing s/v in (12) with Δt_{kp} gives an overall phase factor of $e^{-i\omega\Delta t_{kp}}$. There is an additional electronic delay in the stochastic cooling electronics that will be observed. This electronic delay will be calculated and inserted by hand later. Second, $\beta(s)$ and $\mu(s)$ are evaluated at the pickup. Writing the kicker to pickup phase advance as μ_{kp} and the beta function at the pickup as β_p, the single particle phasor response, $\hat{y}(\omega;\omega_0)$, is given by:

$$\hat{y}(\omega;\omega_0)=\frac{\sqrt{\beta_k\beta_p}}{4}\hat{\theta}e^{-i\omega\Delta t_{kp}}\left\{\frac{e^{-i\left[\mu_{kp}-\pi\left(\frac{\omega}{\omega_0}+Q\right)\right]}}{\sin\pi\left(\frac{\omega}{\omega_0}+Q\right)}-\frac{e^{i\left[\mu_{kp}+\pi\left(\frac{\omega}{\omega_0}-Q\right)\right]}}{\sin\pi\left(\frac{\omega}{\omega_0}-Q\right)}\right\} \tag{13}$$

Three additional modifications are required to make the single particle amplitude given in equation (13) reflect what is actually observed in a beam transfer function measurement.

(1) Transverse stochastic cooling pickups detect the dipole moment of the beam. The dipole moment is the product of the transverse displacement from the center of the pickup and the electric current. The dipole moment phasor, $\hat{d}(\omega;\omega_0)$, of a particle of charge e with the amplitude given in equation (13) is

$$\hat{d}(\omega;\omega_0) = \frac{e\omega_0}{2\pi}\hat{y}(\omega;\omega_0) \tag{14}$$

(2) The second modification is to add the additional time delay in the stochastic cooling system electronics, Δt_e, to Δt_{kp}. In so doing, the notation of van der Meer [2] will be adopted. Define the quantity α to be the fraction of the accelerator circumference which the beam traverses going from pickup to kicker. If the average radius is R_{ave}, then α is given by:

$$\alpha = 1 - \frac{s_p}{2\pi R_{ave}} \tag{15}$$

Thus, Δt_{kp} is can be written in terms of α as:

$$\Delta t_{kp} = (1-\alpha)\frac{2\pi}{\omega_0} \tag{16}$$

The cooling system is phased so that the value of the electronic delay, Δt_e, is equal to the beam transit time from the pickup to the kicker, Δt_{pk}, (so that the kicker corrects the same beam sample the pickup detects). The system is only exactly phased for one revolution frequency. If ω_c is the frequency for which the phasing is exact, the stochastic cooling system electronic delay, Δt_e, is given by:

$$\Delta t_e = \frac{2\pi\alpha}{\omega_c} \tag{17}$$

Thus, Δt_{kp} in equation (13) is to be replaced by the total delay, Δt (electronic + kicker to pickup). The total delay, from equations (16) and (17), is given by:

$$\begin{aligned}\Delta t &= \Delta t_{kp} + \Delta t_e \\ &= 2\pi\left(\frac{1-\alpha}{\omega_0} + \frac{\alpha}{\omega_c}\right) \\ &= \frac{2\pi}{\omega_0} - 2\pi\alpha\left(\frac{1}{\omega_0} - \frac{1}{\omega_c}\right)\end{aligned} \tag{18}$$

Again following van der Meer, the time delay factor, $C(\omega;\omega_0)$ is defined as:

$$C(\omega;\omega_0) = e^{2\pi i\alpha\omega\left(\frac{1}{\omega_0} - \frac{1}{\omega_c}\right)} \tag{19}$$

Therefore, the factor, $e^{-i\omega\Delta t_{kp}}$ in equation (13) is replaced by:

$$e^{-i\omega\Delta t} = C(\omega;\omega_0)\,e^{-2\pi i\frac{\omega}{\omega_0}} \tag{20}$$

(3) The third modification to equation (13) is to express the betatron phase advance from kicker to pickup, μ_{kp}, in terms of the phase advance from pickup to kicker, μ_{pk}. μ_{kp} and μ_{pk} differ by the betatron tune: $\mu_{pk} = 2\pi Q - \mu_{kp}$. Following the notation used by van der Meer, μ_{pk} is written in terms of the quantity α_2, which is defined as the pickup to kicker phase advance expressed as a fraction of the tune. α_2 is given by:

$$\alpha_2 \equiv \frac{\mu_{pk}}{2\pi Q} = 1 - \frac{\mu_{kp}}{2\pi Q} \tag{21}$$

With these three modifications, the single particle dipole phasor amplitude is given by:

$$\hat{d}(\omega;\omega_0)=\frac{e\sqrt{\beta_k\beta_p}}{8\pi}\hat{\theta}\omega_0 C(\omega;\omega_0)\left[\frac{e^{2\pi i\alpha_2 Q}}{\tan\pi\left(\frac{\omega}{\omega_0}+Q\right)}-\frac{e^{-2\pi i\alpha_2 Q}}{\tan\pi\left(\frac{\omega}{\omega_0}-Q\right)}+2\sin 2\pi\alpha_2 Q\right] \quad (22)$$

III Derivation of the Beam Transfer Function

The signal on the pickup is a superposition of the transverse motion of all of the beam particles comprising the momentum distribution of the beam. The average response, $\langle\hat{d}(\omega)\rangle$, is what is observed at the pickup. The average is computed by integrating equation (18) over the tune (Q) and revolution frequency (ω_0) distribution of the beam. At the microwave frequencies of interest for stochastic cooling systems, the tune distribution can be treated as a delta function at the value of the average tune. That is to say, the spread in the beam distribution due to the tune spread is negligible at microwave frequencies in comparison to the spread due to the momentum width of the beam. Therefore, the distribution of interest is the revolution frequency distribution, $\psi(\omega_0)$. $\langle\hat{d}(\omega)\rangle$ is then given by:

$$\left\langle\hat{d}(\omega)\right\rangle=\int_{-\infty}^{\infty}\hat{d}(\omega;\omega_0)\psi(\omega_0)d\omega_0 \quad (23)$$

$\psi(\omega_0)$ is assumed to be normalized such that $\int_{-\infty}^{\infty}\psi(\omega_0)d\omega_0=1$. Substitution of equation (22) for $\hat{d}(\omega)$ in equation (23) gives:

$$\left\langle\hat{d}(\omega)\right\rangle=\frac{e\sqrt{\beta_k\beta_p}}{8\pi}\hat{\theta}\left\{\begin{array}{l} e^{2\pi i\alpha_2 Q}\displaystyle\int_{-\infty}^{\infty}\frac{\omega_0 C(\omega;\omega_0)\psi(\omega_0)}{\tan\pi\left(\frac{\omega}{\omega_0}+Q\right)}d\omega_0 \\ -\;e^{-2\pi i\alpha_2 Q}\displaystyle\int_{-\infty}^{\infty}\frac{\omega_0 C(\omega;\omega_0)\psi(\omega_0)}{\tan\pi\left(\frac{\omega}{\omega_0}-Q\right)}d\omega_0 \\ +\;2\sin 2\pi\alpha_2 Q\displaystyle\int_{-\infty}^{\infty}\omega_0 C(\omega;\omega_0)\psi(\omega_0)d\omega_0 \end{array}\right\} \quad (24)$$

Evaluation of equation (24) requires the evaluation of the following integrals:

$$I_{\pm}(\omega)=\int_{-\infty}^{\infty}\frac{\omega_0 C(\omega;\omega_0)}{\tan\pi\left(\frac{\omega}{\omega_0}\pm q\right)}\psi(\omega_0)d\omega_0 \quad (25)$$

Here, "+" indicates the slow-wave term and "–" indicates the fast-wave term. q is the fractional part of the betatron tune. Since the tangent function has period π, only the fractional part of the tune is relevant[a].

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[a] Note: This statement is not true of the functions with argument $\pm 2\pi\alpha_2 Q$ in equation (24).

For any integer, n, the denominator in the integrand of equation (25) vanishes at $\omega_0 = \omega/(n \mp q)$. It is apparent that the betatron lower sideband resonances – $\omega = (n-q)\omega_0$ – come from the slow-wave term and that the upper sideband resonances – $\omega = (n+q)\omega_0$ – come from the fast-wave term. The value of n used in the slow-wave term is not necessarily the same as that used in the fast-wave term. Since we are interested in overlapping adjacent sidebands, a suitable choice would be n for the slow-wave term and $n-1$ for the fast-wave term. Figure 2 shows the arrangement of the upper and lower sidebands in the betatron spectrum of the beam.

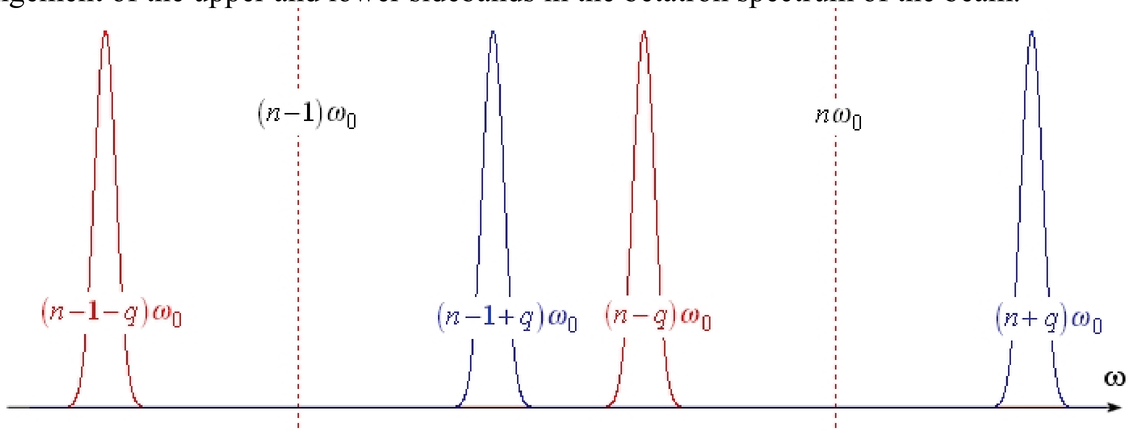

Figure 2 This figure shows a portion of the transverse schottky spectrum of the beam. The arrangement of the upper and lower betatron sidebands and their relationship to harmonics of the revolution frequency is illustrated. This arrangement is valid for $q < 0.5$.

The integrals of equation (25) can be evaluated by contour integration along a path that traverses the real axis from $-\infty$ to $+\infty$ and is assumed to close in a way that makes the off-axis portion of the contour integral zero. The integration along the real axis gives a Cauchy principle value integral plus a contribution from the residues at the poles of the integrand. Equation (25) becomes:

$$I_{\pm}(\omega) = \mathrm{PV}\int_{-\infty}^{\infty} \frac{\omega_0 C(\omega;\omega_0)}{\tan \pi\left(\dfrac{\omega}{\omega_0} \pm q\right)} \psi(\omega_0)\, d\omega_0 - i\pi \sum_n \mathrm{Res}\left(\frac{\omega}{n \mp q}\right) \tag{26}$$

The sign of the residue term is determined by whether or not the integration contour is chosen to go above or below the real axis around the pole. The minus sign chosen here reflects the fact that the contour was deformed so that it goes above the real axis at the pole. This is equivalent to adding a small negative imaginary part to ω. From equation (1), it is evident that this corresponds to an excitation that was zero in the distant past ($t \rightarrow -\infty$). The alternative route around the pole corresponds to and infinite excitation in the distant past, which is not physically meaningful.

The poles are of order 1. Therefore, the residue can be calculated in the following way [3]:

$$\mathrm{Res}\left(\frac{\omega}{n \mp q}\right) = \frac{\left(\dfrac{\omega}{n \mp q}\right) C\left(\omega; \dfrac{\omega}{n \mp q}\right) \psi\left(\dfrac{\omega}{n \mp q}\right)}{\left.\dfrac{d}{d\omega_0} \tan \pi\left(\dfrac{\omega}{\omega_0} \pm q\right)\right|_{\omega_0 = \frac{\omega}{n \mp q}}} \tag{27}$$

This evaluates to:

$$\mathrm{Res}\left(\frac{\omega}{n\mp q}\right)=-\frac{1}{\pi}\frac{\omega^2}{\left(n\mp q\right)^3}C\left(\omega;\frac{\omega}{n\mp q}\right)\psi\left(\frac{\omega}{n\mp q}\right) \tag{28}$$

Making the change of variable $\omega_\beta=(n\mp q)\omega_0$ and substituting from (28) for the residue term, the integrals of equation (26) become[b]:

$$I_\pm(\omega)=\frac{1}{\left(n\mp q\right)^2}\mathrm{PV}\int_{-\infty}^{\infty}\frac{\omega_\beta C\left(\omega;\frac{\omega_\beta}{n\mp q}\right)\psi\left(\frac{\omega_\beta}{n\mp q}\right)}{\tan\left[\frac{\pi(n\mp q)}{\omega_\beta}\left(\omega-\omega_\beta\right)\right]}d\omega_\beta+i\sum_n\frac{\omega^2}{\left(n\mp q\right)^3}C\left(\omega;\frac{\omega}{n\mp q}\right)\psi\left(\frac{\omega}{n\mp q}\right) \tag{29}$$

If the betatron sideband excitation is entirely contained in a width that is less than ½ω_0, the sum over residues will only include one term – the $n-q$ term for I_+ and the $n-1+q$ term for I_-. Since I_+ is associated with the lower sideband resonances and I_- with the upper sideband resonances, the principle value integrals in equation (29) will be labeled accordingly. The upper and lower side band principal value integrals are:

$$\begin{aligned}I_{LSB}(\omega)&=\frac{1}{\left(n-q\right)^2}\mathrm{PV}\int_{-\infty}^{\infty}\frac{\omega_\beta C\left(\omega;\frac{\omega_\beta}{n-q}\right)\psi\left(\frac{\omega_\beta}{n-q}\right)}{\tan\left[\frac{\pi(n-q)}{\omega_\beta}\left(\omega-\omega_\beta\right)\right]}d\omega_\beta\\ I_{USB}(\omega)&=\frac{1}{\left(n-1+q\right)^2}\mathrm{PV}\int_{-\infty}^{\infty}\frac{\omega_\beta C\left(\omega;\frac{\omega_\beta}{n-1+q}\right)\psi\left(\frac{\omega_\beta}{n-1+q}\right)}{\tan\left[\frac{\pi(n-1+q)}{\omega_\beta}\left(\omega-\omega_\beta\right)\right]}d\omega_\beta\end{aligned} \tag{30}$$

The evaluation of these integrals is dealt with in Appendix B.

Combining equations (24), (29), and (30), gives the average response of the beam, $\langle\hat{d}(\omega)\rangle$:

[b] The argument of the tangent function in the denominator of the integrand is manipulated as follows:

$$\begin{aligned}\tan\pi\left(\frac{\omega}{\omega_0}\mp q\right)&=\tan\pi\left[\frac{\left(n\mp q\right)\omega}{\omega_\beta}\pm q\right]=\tan\pi\left[\frac{\left(n\mp q\right)\omega}{\omega_\beta}\pm q-n\right]\\&=\tan\left[\pi\frac{\left(n\mp q\right)}{\omega_\beta}\left(\omega-\omega_\beta\right)\right]\end{aligned}$$

$$\langle\hat{d}(\omega)\rangle = \frac{e\sqrt{\beta_k\beta_p}}{8\pi}\hat{\theta}\left\{\begin{array}{l} e^{2\pi i\alpha_2 Q}\left[I_{LSB}(\omega)+\dfrac{i\omega^2}{(n-q)^3}C\left(\omega;\dfrac{\omega}{n-q}\right)\psi\left(\dfrac{\omega}{n-q}\right)\right] \\ -\ e^{-2\pi i\alpha_2 Q}\left[I_{USB}(\omega)+\dfrac{i\omega^2}{(n-1+q)^3}C\left(\omega;\dfrac{\omega}{n-1+q}\right)\psi\left(\dfrac{\omega}{n-1+q}\right)\right] \\ +\ 2\sin(2\pi\alpha_2 Q)\int_{-\infty}^{\infty}\omega_0 C(\omega;\omega_0)\psi(\omega_0)\,d\omega_0 \end{array}\right\} \tag{31}$$

It is interesting to note that there is a non-resonant component to the response. The non-resonant component is the $2\sin(2\pi\alpha_2 Q)$ term. This term gives the closed orbit distortion that arises from the dipole kick to the beam. The fact that this term is physically meaningful is confirmed by setting ω equal to 0 in equation (31). With $\omega = 0$, $C(0;\ \omega_0) = 1$, and the residue terms vanish.

The dc evaluation of equation (31) is:

$$\langle\hat{d}(0)\rangle = \frac{e\sqrt{\beta_k\beta_p}}{8\pi}\hat{\theta}\left[e^{2\pi i\alpha_2 Q}I_{LSB}(0) - e^{-2\pi i\alpha_2 Q}I_{USB}(0) + 2\sin(2\pi\alpha_2 Q)\langle\omega_0\rangle\right] \tag{32}$$

where $\langle\omega_0\rangle = \int_{-\infty}^{\infty}\omega_0\psi(\omega_0)\,d\omega_0$, and $I_{LSB}(0)$ and $I_{USB}(0)$ are given by:

$$\begin{aligned} I_{LSB}(0) &= \frac{\cot\pi q}{(n-q)^2}\int_{-\infty}^{\infty}\omega_\beta\psi\left(\frac{\omega_\beta}{n-q}\right)d\omega_\beta \\ &= \cot\pi q\int_{-\infty}^{\infty}\omega_0\psi(\omega_0)\,d\omega_0 \\ &= \cot\pi q\langle\omega_0\rangle = -I_{USB}(0) \end{aligned} \tag{33}$$

Putting this all together gives:

$$\begin{aligned} \langle\hat{d}(0)\rangle &= \frac{e\sqrt{\beta_k\beta_p}}{4\pi}\hat{\theta}\langle\omega_0\rangle\left(\cot\pi q\cos 2\pi\alpha_2 Q + \sin 2\pi\alpha_2 Q\right) \\ &= \frac{e\sqrt{\beta_k\beta_p}}{4\pi\sin\pi q}\hat{\theta}\langle\omega_0\rangle\cos\pi(q-2\alpha_2 Q) \\ &= \frac{e\sqrt{\beta_k\beta_p}}{4\pi\sin\pi q}\hat{\theta}\langle\omega_0\rangle\cos\left(\pi q - \mu_{kp}\right) \end{aligned} \tag{34}$$

The last line follows from the definition of α_2 given in equation (21). Using equation (14) to write this in terms of $\langle\hat{y}\rangle$ gives:

$$\langle\hat{y}\rangle = \frac{\sqrt{\beta_k\beta_p}}{2\sin\pi q}\hat{\theta}\cos\left(\pi q - \mu_{kp}\right) \tag{35}$$

This is the well-known formula for the closed orbit deviation at a betatron phase advance of μ_{kp} from a single dipole kick.

The quantity of interest for stochastic cooling calculations is the beam transfer function, $B(\omega)$. $B(\omega)$ is defined to be the average beam dipole moment response per unit dipole kick. From equation (31) the beam transfer function is given by:

$$B(\omega) \equiv \frac{\left\langle \hat{d}(\omega) \right\rangle}{\hat{\theta}}$$

$$= \frac{e\sqrt{\beta_k \beta_p}}{8\pi} \left\{ \begin{array}{l} e^{2\pi i \alpha_2 Q} \left[I_{LSB}(\omega) + \frac{i\omega^2}{(n-q)^3} C\left(\omega; \frac{\omega}{n-q}\right) \psi\left(\frac{\omega}{n-q}\right) \right] \\ - \; e^{-2\pi i \alpha_2 Q} \left[I_{USB}(\omega) + \frac{i\omega^2}{(n-1+q)^3} C\left(\omega; \frac{\omega}{n-1+q}\right) \psi\left(\frac{\omega}{n-1+q}\right) \right] \\ + \; 2\sin(2\pi\alpha_2 Q) \int_{-\infty}^{\infty} \omega_0 C(\omega; \omega_0)\, \psi(\omega_0)\, d\omega_0 \end{array} \right\} \tag{36}$$

IV. Application to the Recycler Ring

The expression for $B(\omega)$ in equation (36) will now be applied to the 2-4 GHz vertical stochastic cooling system in the Recycler ring. Table 1 lists the Recycler parameters used in this calculation.

Table 1 Recycler Parameters

Parameter	Symbol	Value
Revolution Frequency	f_{rev}	89,813.28 kHz
Beam Momentum	p	8878.8 MeV/c
Circumference	L	3319.3980 m
Slip factor	η	-0.00898
Vertical Tune	Q	24.443
Kicker to Pickup distance	s_p	2677.3084 m
Pickup to Kicker fraction of L	α	0.193436
Pickup to Kicker fraction of Q	α_2	0.174066
Beta function at Pickup	β_p	25.6304 m
Beta function at Kicker	β_k	19.0310 m

A gaussian revolution frequency distribution was used for this calculation. The beam widths used will be specified as momentum spreads. The revolution frequency width, σ_f, is related to the momentum spread, σ_p, by:

$$\frac{\sigma_f}{f_{rev}} = \eta \frac{\sigma_p}{p} \tag{37}$$

The frequency of the upper and lower sidebands, f_β, is given by:

$$f_\beta = (n \pm q) f_{rev} \tag{38}$$

The frequency distribution of the betatron sidebands is obtained from the revolution frequency distribution in the following way:

$$\psi\left(f_\beta\right)df_\beta = \psi\left(f_{rev}\left(f_\beta\right)\right)\frac{df_{rev}}{df_\beta}df_\beta$$
$$= \frac{1}{n\pm q}\psi\left(\frac{f_{rev}}{n\pm q}\right)df_\beta \tag{39}$$

For a gaussian distribution, $\psi(f_\beta)$ becomes:

$$\psi\left(f_\beta\right) = \frac{1}{\sqrt{2\pi}\left(n\pm q\right)\sigma_f}e^{-\frac{1}{2}\left[\frac{f_\beta-(n\pm q)f_{ave}}{(n\pm q)\sigma_f}\right]^2} \tag{40}$$

f_{ave} is the average revolution frequency.

Inspection of equation (40) reveals that the width of the f_β distribution is given by $\sigma_\beta = \left(n\pm q\right)\sigma_f$. Thus, the betatron sidebands get wider with increasing harmonic number. Consequently, at some harmonic number, the sidebands will begin to overlap. For $q < 0.5$, the harmonic number at which the upper and lower betatron sidebands are within one σ_β of each other is given by:

$$\left(n-q\right)f_{rev} - \left(n-1+q\right)f_{rev} < \sigma_\beta = \left(n-q\right)\sigma_f = \frac{\left(n-q\right)\eta f_{rev}}{p}\sigma_p$$
$$1-2q < \frac{n-q}{p}\eta\sigma_p \tag{41}$$
$$n > q + \left(1-2q\right)\frac{p}{\eta\sigma_p}$$

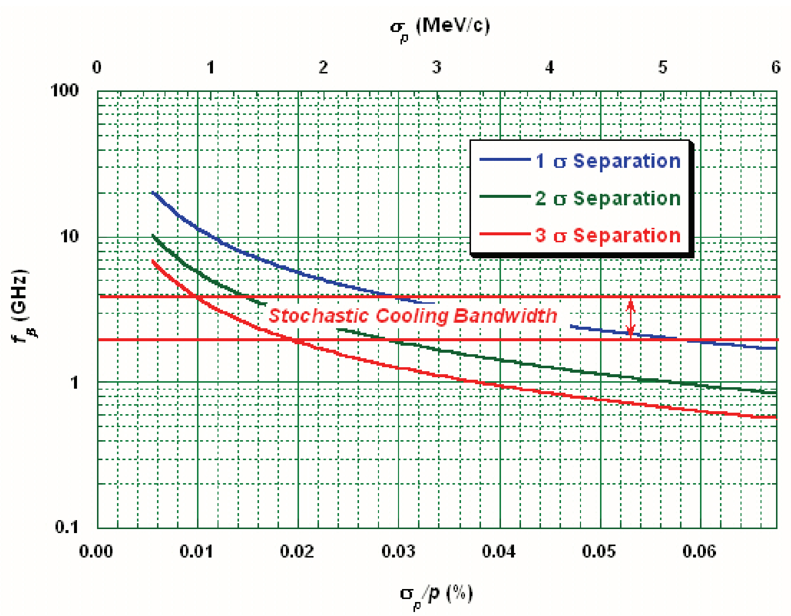

Figure 3 Lower sideband frequency at which adjacent upper and lower sidebands in the Recycler are separated by one, two, and three betatron σ as a function of beam momentum spread.

Equation (41) shows that the harmonic at which sideband overlap begins to occur is proportional to the proximity of the tune to ½. Figure 3 shows the relationship between the lower sideband frequency at which overlap begins and σ_p. It is clear from this figure, that beams with a σ_p greater than about 2 MeV/c there will be some overlap of upper and lower sidebands for the entire 2-4 GHz bandwidth of the Recycler transverse cooling.

Figure 4 shows the vertical beam transfer function at 3.2 GHz for beam with σ_p = 1.0 MeV/c. There are two peaks in the response – one that corresponds to an upper sideband resonance and one corresponding to a lower sideband resonance.

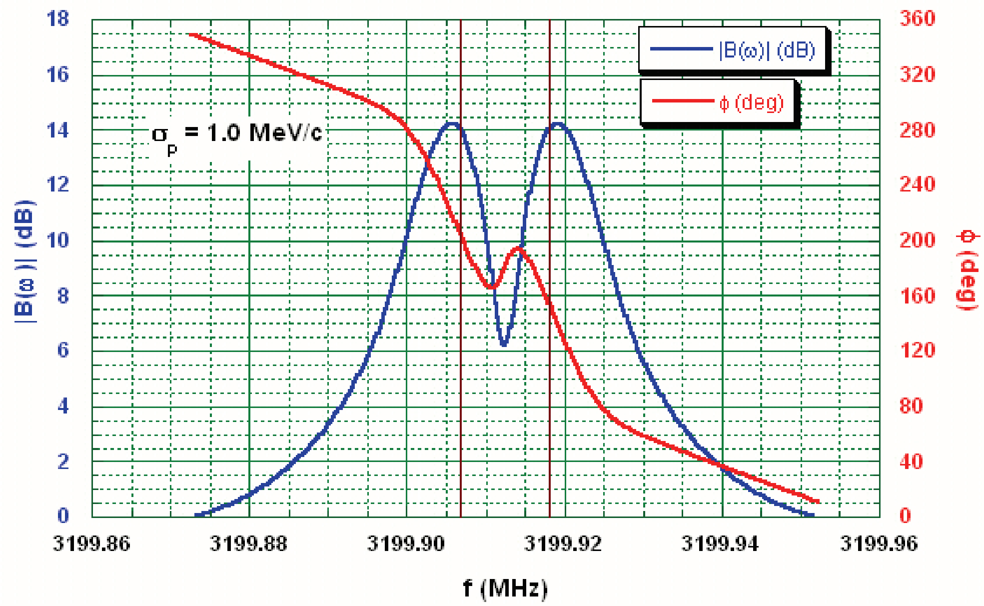

Figure 4 Vertical beam transfer function measurement at 3.2 GHz for beam with σ_p = 1 MeV/c. The left peak corresponds to the upper side band resonance $(35628 + q)f_{rev}$, and the right peak corresponds to the lower sideband resonance $(35629 - q)f_{rev}$. The brown markers indicate the exact location of these frequencies.

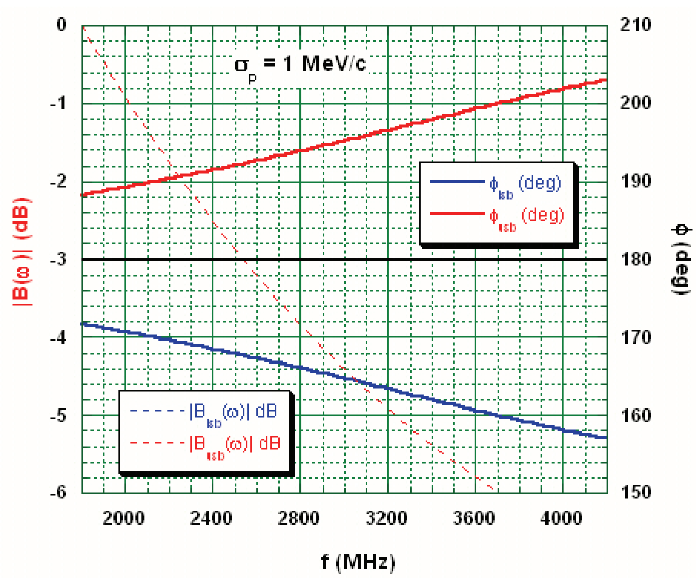

Figure 5 The variation of amplitude and phase at the vertical upper and lower sideband frequencies across the bandwidth of the stochastic cooling system. The amplitude variation for upper and lower sideband are identical, therefore their graphs are not distinguishable.

The interference between the two sidebands causes the peak response to move away from the resonant frequency and to move the phase at the resonant frequencies away from 180º. The magnitude of these effects increases with increasing σ_p and as the frequency is increased (see equation (41) and Figure 5).

If σ_p is increased to 2 MeV/c, the individual sidebands are no longer distinguishable (see Figure 6). The variation of the phase measured at the sideband frequencies also changes with increased σ_p (see Figure 7).

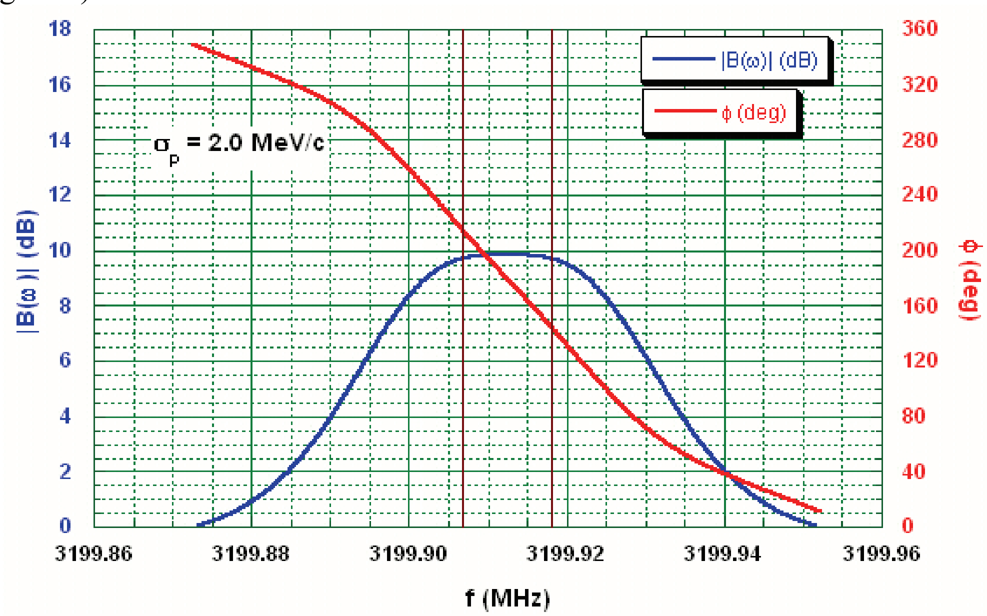

Figure 6 Vertical beam transfer function measurement at 3.2 GHz for beam with σ_p = 2 MeV/c. The upper and lower sideband peaks cannot be distinguished from one another. The brown markers indicate the exact location of the sideband frequencies.

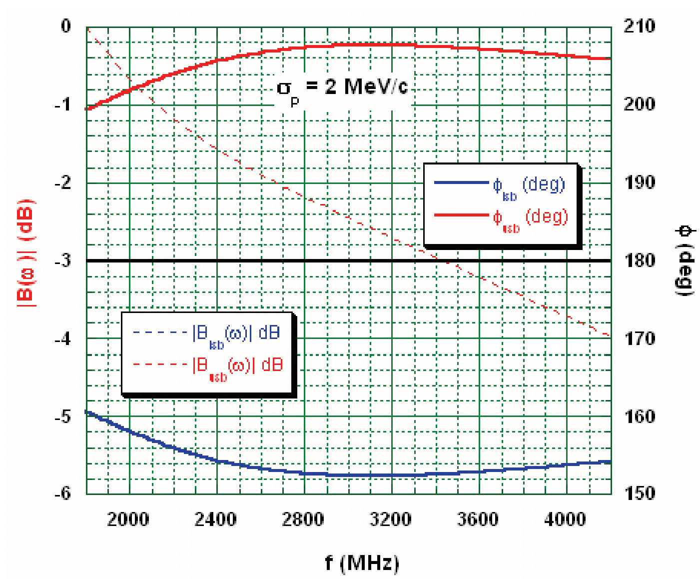

Figure 7 The variation of amplitude and phase at the vertical upper and lower sideband frequencies across the bandwidth of the stochastic cooling system. The amplitude variation for upper and lower sideband are identical, therefore their graphs are not distinguishable.

V. References

1 S. van der Meer, A Different Formulation of the Longitudinal and Transverse Beam Response, CERN/PS/AA/80-4, 1980.

2 S. van der Meer, *op. cit.* equations 10, 18, and 41.

3 E.B. Saff & A.D. Snider, Fundamentals of Complex Analysis, Prentice-Hall, 1976, page 240.

Appendix A

Determination of the time dependent solutions to Hill's equation for a sinusoidal excitation

For those who possess a perverse desire to know all of the details but don't derive any satisfaction from working them out for themselves, here is the derivation of equation (8) above. First, it will be shown that equation (8) is in fact a solution of the time dependent problem. The boundary conditions, given in equations (3), will then be used to prove that equation (7) is the particular solution for the beam transfer function problem.

1. Hill's Equation

The time independent form of Hill's equation is:

$$\frac{d^2 y}{ds^2} + K(s)y = 0 \tag{A-1}$$

The time dependant form is obtained by replacing $\frac{dy}{ds}$ with:

$$\frac{dy(s,t)}{ds} = \frac{\partial y(s,t)}{\partial s} + \frac{1}{v}\frac{\partial y(s,t)}{\partial t} \tag{A-2}$$

The second derivative is obtained by differentiation of equation (A-2). The result is:

$$\frac{d^2 y}{ds^2} = \frac{\partial^2 y}{\partial s^2} + \frac{2}{v}\frac{\partial^2 y}{\partial s \partial t} + \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{A-3}$$

Substitution of equation (A-3) into equation (A-1) gives Hill's equation in terms of derivatives with respect to the explicit longitudinal and time dependence occurring in the expected solution. The time dependant form of Hill's equation is:

$$\frac{\partial^2 y(s,t)}{\partial s^2} + \frac{2}{v}\frac{\partial^2 y(s,t)}{\partial s \partial t} + \frac{1}{v^2}\frac{\partial^2 y(s,t)}{\partial t^2} + K(s)y(s,t) = 0 \tag{A-4}$$

2. Derivatives of y(s,t)

The proposed solutions of (A-4) are:

$$y(s,t) = \sqrt{\varepsilon\beta(s)}\left\{A_+(\omega)e^{i\left[\mu(s)+\omega\left(t-\frac{s}{v}\right)\right]} + A_-(\omega)e^{-i\left[\mu(s)-\omega\left(t-\frac{s}{v}\right)\right]}\right\} \tag{A-5}$$

The notation is somewhat simplified by working with $y(s, t)$ expressed in the following way:

$$y(s,t) = A_+(\omega)y_+(s)e^{i\omega\left(t-\frac{s}{v}\right)} + A_-(\omega)y_-(s)e^{i\omega\left(t-\frac{s}{v}\right)} \tag{A-6}$$

where the $y_\pm(s)$ are given by:

$$y_\pm(s) = \sqrt{\varepsilon\beta(s)}\,e^{\pm i\mu(s)} \tag{A-7}$$

The constants $A_+(\omega)$ and $A_-(\omega)$ are obtained by repeated differentiation of equation (A-6) with respect to s and t, substitution of these results into (A-4), and applying the boundary conditions. The various derivatives of $y(s, t)$ are:

$$\frac{\partial y(s,t)}{\partial s} = -i\frac{\omega}{v}y(s,t) + \left[A_+(\omega)y'_+(s) + A_-(\omega)y'_-(s)\right]e^{i\omega\left(t-\frac{s}{v}\right)}$$

$$\frac{\partial y(s,t)}{\partial t} = i\omega y(s,t)$$

$$\frac{\partial^2 y(s,t)}{\partial s^2} = -i\frac{\omega}{v}\left\{-i\frac{\omega}{v}y(s,t) + \left[A_+(\omega)y'_+(s) + A_-(\omega)y'_-(s)\right]e^{i\omega\left(t-\frac{s}{v}\right)}\right\}$$

$$+\left[A_+(\omega)y''_+(s) + A_-(\omega)y''_-(s)\right]e^{i\omega\left(t-\frac{s}{v}\right)}$$

$$-i\frac{\omega}{v}\left[A_+(\omega)y'_+(s) + A_-(\omega)y'_-(s)\right]e^{i\omega\left(t-\frac{s}{v}\right)}$$

$$= -\left[\left(\frac{\omega}{v}\right)^2 + K(s)\right]y(s,t) - 2i\frac{\omega}{v}\left[A_+(\omega)y'_+(s) + A_-(\omega)y'_-(s)\right]e^{i\omega\left(t-\frac{s}{v}\right)}$$

$$\frac{2}{v}\frac{\partial^2 y(s,t)}{\partial s \partial t} = 2\left(\frac{\omega}{v}\right)^2 y(s,t) + 2i\frac{\omega}{v}\left[A_+(\omega)y'_+(s) + A_-(\omega)y'_-(s)\right]e^{i\omega\left(t-\frac{s}{v}\right)} \tag{A-8}$$

$$\frac{1}{v^2}\frac{\partial^2 y(s,t)}{\partial t^2} = -\left(\frac{\omega}{v}\right)^2 y(s,t)$$

It is seen by inspection of equations (A-8) (i.e. adding up the last three lines) that a solution of the form given in equation (A-5) is indeed a solution to the time dependent form of Hill's Equation.

3. Applying the boundary conditions

The quantities $A_+(\omega)$ and $A_-(\omega)$ are determined by application of the boundary conditions given in equations (3). Continuity of *y(s,t)* at the kicker (*s = 0*) at all times gives:

$$A_+(\omega) + A_-(\omega) = \left[A_+(\omega)e^{2\pi iQ} + A_-(\omega)e^{-2\pi iQ}\right]e^{-2\pi i\frac{\omega}{\omega_0}}$$

$$\frac{A_+(\omega)}{A_-(\omega)} = -e^{-2\pi iq}\frac{\sin\pi\left(\frac{\omega}{\omega_0}+q\right)}{\sin\pi\left(\frac{\omega}{\omega_0}-q\right)} \tag{A-9}$$

where $2\pi Q = \mu(L)$, is the total betatron phase advance around the ring, q is the fractional tune in the transverse plane of interest, and $\omega_0 = 2\pi v/L$ is the angular revolution frequency of the beam.

From equation (A-2), the boundary condition on $y'(L,t)$ takes the form:

$$\begin{aligned}\frac{\partial y(0,t)}{\partial s}+i\frac{\omega}{v}y(0,t)&=\frac{\partial y(L,t)}{\partial s}+i\frac{\omega}{v}y(L,t)+\hat{\theta}e^{i\omega t}\\ \frac{\partial y(0,t)}{\partial s}&=\frac{\partial y(L,t)}{\partial s}+\hat{\theta}e^{i\omega t}\end{aligned}\tag{A-10}$$

The last line follows from the continuity of *y(s,t)* at *s = 0*. Applying this result to equation (A-6) gives:

$$A_+(\omega)y_+'(0)+A_-(\omega)y_-'(0)=\left[A_+(\omega)y_+'(L)+A_-(\omega)y_-'(L)\right]e^{-2\pi i\frac{\omega}{\omega_0}}+\hat{\theta}\tag{A-11}$$

The derivatives $y_\pm'(s)$, are obtained from equation (A-7):

$$\begin{aligned}y_\pm'(s)&=\sqrt{\varepsilon}\left[\frac{1}{2}\beta(s)^{-\frac{1}{2}}\beta'(s)\pm i\beta(s)^{\frac{1}{2}}\mu'(s)\right]e^{\pm i\mu(s)}\\ &=\sqrt{\frac{\varepsilon}{\beta(s)}}\left[\frac{1}{2}\beta'(s)\pm i\right]e^{\pm i\mu(s)}\end{aligned}\tag{A-12}$$

At the kicker, $\beta(0)=\beta(L)\equiv\beta_k$, and $\beta'(0)=\beta'(L)\equiv\beta_k'$. Therefore at *s = 0* and at *s = L* equation (A-12) becomes:

$$\begin{aligned}y_\pm'(0)&=\sqrt{\frac{\varepsilon}{\beta_k}}\left(\frac{1}{2}\beta_k'\pm i\right)\equiv C_\pm\\ y_\pm'(L)&=C_\pm e^{\pm 2\pi iq}\end{aligned}\tag{A-13}$$

Substitution of (A-13) into (A-11) gives the boundary condition on $y'(L,t)$:

$$A_+(\omega)C_++A_-(\omega)C_-=A_+(\omega)C_+e^{-2\pi i\left(\frac{\omega}{\omega_0}-q\right)}+A_-(\omega)C_-e^{-2\pi i\left(\frac{\omega}{\omega_0}+q\right)}+\hat{\theta}\tag{A-14}$$

$A_+(\omega)$ and $A_-(\omega)$ are obtained from equations (A-9) and (A-14):

$$\begin{aligned}A_+(\omega)&=-\frac{1}{4}\sqrt{\frac{\beta_k}{\varepsilon}}\hat{\theta}\frac{e^{\pi i\left(\frac{\omega}{\omega_0}-q\right)}}{\sin\pi\left(\frac{\omega}{\omega_0}-q\right)}\\ A_-(\omega)&=\frac{1}{4}\sqrt{\frac{\beta_k}{\varepsilon}}\hat{\theta}\frac{e^{\pi i\left(\frac{\omega}{\omega_0}+q\right)}}{\sin\pi\left(\frac{\omega}{\omega_0}+q\right)}\end{aligned}\tag{A-15}$$

Note that the beta-function derivatives at the kicker have conveniently canceled.

Substitution of (A-15) into equation (A-5) yields:

$$y(s,t)=\frac{\sqrt{\beta_k\beta(s)}}{4}\hat{\theta}\left\{\frac{e^{-i\left[\mu(s)-\pi\left(\frac{\omega}{\omega_0}+Q\right)-\omega\left(t-\frac{s}{v}\right)\right]}}{\sin\pi\left(\frac{\omega}{\omega_0}+Q\right)}-\frac{e^{i\left[\mu(s)+\pi\left(\frac{\omega}{\omega_0}-Q\right)+\omega\left(t-\frac{s}{v}\right)\right]}}{\sin\pi\left(\frac{\omega}{\omega_0}-Q\right)}\right\}\tag{A-16}$$

This is the beam response to a periodic dipole kick – also given in equation (8) in the text.

Appendix B

Evaluation of the principle value integrals of Equation (30)

The evaluation of the principle value integrals in equation (30) requires a bit of explanation. The integral of interest is of the form:

$$I(\omega) = PV \int_{-\infty}^{\infty} \frac{\omega_\beta C\left(\omega; \frac{\omega_\beta}{n \pm q}\right) \psi\left(\frac{\omega_\beta}{n \pm q}\right)}{\tan\left[\frac{\pi(n \pm q)}{\omega_\beta}(\omega - \omega_\beta)\right]} d\omega_\beta \tag{B-1}$$

$I(\omega)$ can be split into three parts as follows:

$$I_1(\omega) = \left(\frac{\omega_0}{\omega_r}\right)^2 \int_{-\infty}^{\omega - \delta\omega} \frac{f(\omega; \omega_\beta)}{\tan\left(\frac{\pi\omega_r}{\omega_0} \frac{\omega - \omega_\beta}{\omega_\beta}\right)} d\omega_\beta \tag{B-2}$$

$$I_2(\omega) = \left(\frac{\omega_0}{\omega_r}\right)^2 PV \int_{\omega - \delta\omega}^{\omega + \delta\omega} \frac{f(\omega; \omega_\beta)}{\tan\left(\frac{\pi\omega_r}{\omega_0} \frac{\omega - \omega_\beta}{\omega_\beta}\right)} d\omega_\beta \tag{B-3}$$

$$I_3(\omega) = \left(\frac{\omega_0}{\omega_r}\right)^2 \int_{\omega + \delta\omega}^{\infty} \frac{f(\omega; \omega_\beta)}{\tan\left(\frac{\pi\omega_r}{\omega_0} \frac{\omega - \omega_\beta}{\omega_\beta}\right)} d\omega_\beta \tag{B-4}$$

Where $f(\omega; \omega_\beta) = \omega_\beta C\left(\omega; \frac{\omega_\beta}{n \pm q}\right) \psi\left(\frac{\omega_\beta}{n \pm q}\right)$ and $\omega_r \equiv (n \pm q)\omega_0$. $\delta\omega$ is chosen to be small enough that a linear approximation to the tangent function in the denominator is feasible in the neighborhood of $\omega = \omega_\beta$.

I_1 and I_3 are non-singular and can be integrated numerically with little trouble. I_2 is singular at $\omega = \omega_\beta$; therefore this part of the integration must be handled with a little more care.

Evaluation of $I_2(\omega)$

Thus, with a suitably small $\delta\omega$, I_2, in equation (B-3), can be approximated by:

$$I_2(\omega) \simeq \frac{1}{\pi}\left(\frac{\omega_0}{\omega_r}\right)^3 PV \int_{\omega - \delta\omega}^{\omega + \delta\omega} \frac{\omega_\beta}{\omega - \omega_\beta} f(\omega; \omega_\beta)\, d\omega_\beta \tag{B-5}$$

The integration of (B-5) is simplified by use of the following device to mitigate the singularity at $\omega = \omega_\beta$:

$$\int_{a-\varepsilon}^{a+\varepsilon} \frac{f(x)}{a-x} dx = \int_{a-\varepsilon}^{a+\varepsilon} \frac{f(x)-f(a)}{a-x} dx + \int_{a-\varepsilon}^{a+\varepsilon} \frac{f(a)}{a-x} dx$$
$$= \int_{a-\varepsilon}^{a+\varepsilon} \frac{f(x)-f(a)}{a-x} dx$$

Thus, I_2 can be written as:

$$I_2(\omega) \simeq \frac{1}{\pi}\left(\frac{\omega_0}{\omega_r}\right)^3 PV \int_{\omega-\delta\omega}^{\omega+\delta\omega} \frac{\omega_\beta f(\omega;\omega_\beta) - \omega f(\omega;\omega)}{\omega - \omega_\beta} d\omega_\beta \tag{B-6}$$

Determination of $\delta\omega$

For small x, $\tan^{-1} x$ can be approximated by:

$$\frac{1}{\tan x} = \cot x \simeq \frac{1}{x} - \frac{x}{3} + \mathcal{O}(x^3) \tag{B-7}$$

If $x < 1$, the higher order terms will all be smaller than $\frac{x}{3}$.

The remainder for the series expansion of cot x is difficult to calculate. A non-rigorous shortcut is to require the $\frac{x}{3}$ term in equation (B-7) to be less than some small number δ. δ is comparable to the required precision of the calculation. In the present case, x in equation (B-7) is given by:

$$x = \frac{\pi\omega_r}{\omega_0}\frac{\omega - \omega_\beta}{\omega_\beta} \tag{B-8}$$

Thus, $\delta\omega$ is determined from:

$$I_\delta \equiv \left| \frac{\pi}{3}\frac{\omega_0}{\omega_r} \int_{\omega-\delta\omega}^{\omega+\delta\omega} \frac{\omega - \omega_\beta}{\omega_\beta} f(\omega;\omega_\beta) d\omega_\beta \right| < \delta \tag{B-9}$$

With a change of variable, $x = \omega - \omega_\beta$, I_δ becomes:

$$\begin{aligned} I_\delta &= \left| \frac{\pi}{3}\frac{\omega_0}{\omega_r} \int_{-\delta\omega}^{\delta\omega} \frac{x}{\omega - x} f(\omega;\omega - x) dx \right| \\ &= \left| \frac{\pi}{3}\frac{\omega_0}{\omega_r} \int_{-\delta\omega}^{\delta\omega} x C\left(\omega; \frac{\omega_0}{\omega_r}(\omega - x)\right) \psi\left(\frac{\omega_0}{\omega_r}(\omega - x)\right) dx \right| \end{aligned} \tag{B-10}$$

Finally, $\delta\omega$ is determined by solving:

$$\left| \frac{\pi}{3}\frac{\omega_0}{\omega_r} \int_{-\delta\omega}^{\delta\omega} x C\left(\omega; \frac{\omega_0}{\omega_r}(\omega - x)\right) \psi\left(\frac{\omega_0}{\omega_r}(\omega - x)\right) dx \right| < \delta \tag{B-11}$$

It should be noted that the value of $\delta\omega$ that solves equation (B-11) is not the same for all ω. In general $\delta\omega$ is a function of ω. Near the betatron resonance frequencies $\delta\omega$ will need to be smaller to maintain the same precision in the I_2 integration. Figure B- 1 shows a typical result for the ω dependence of $\delta\omega$.

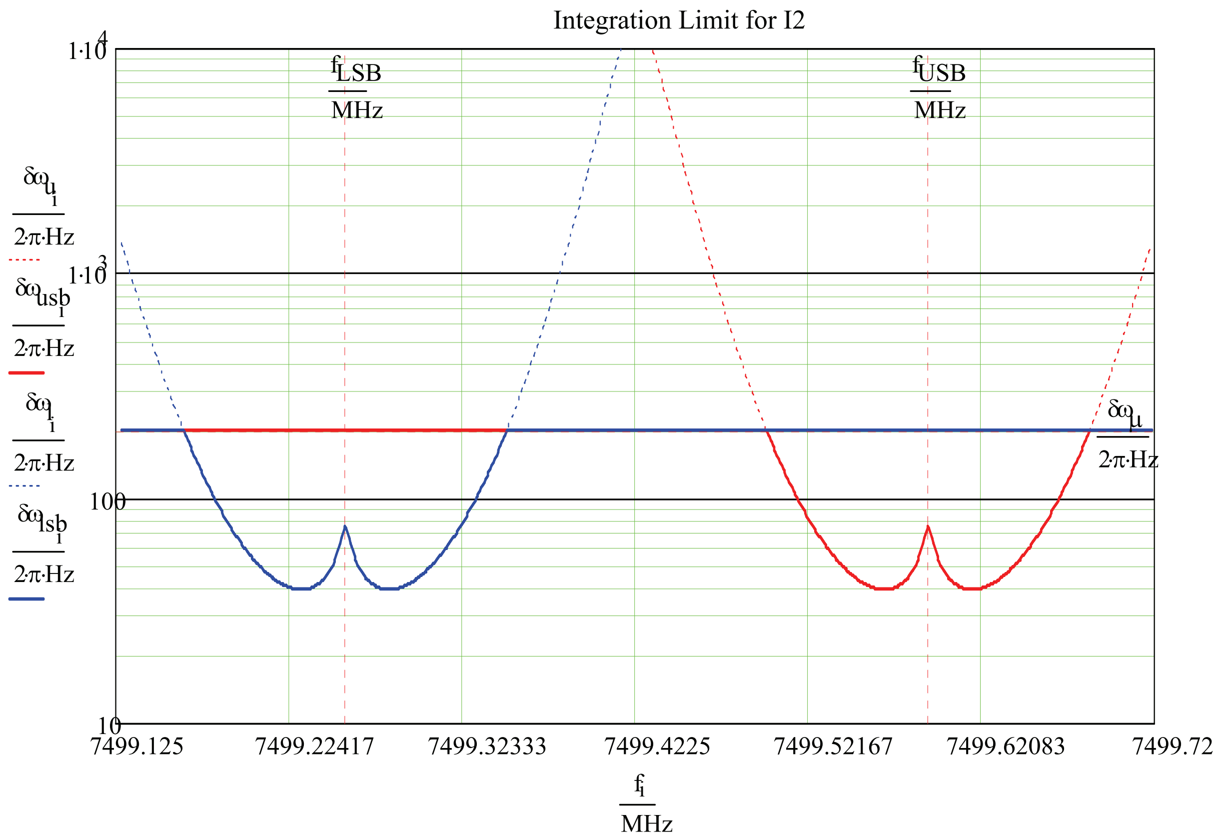

Figure B- 1 Plot of $\delta\omega$ for upper ($\delta\omega_u$) and lower($\delta\omega_u$) sideband integrations. This plot comes from a calculation of sideband overlap for the Debuncher vertical band 4 cooling system. The frequencies of the upper and lower sideband resonances are labeled f_{USB} and f_{LSB} respectively. The maximum $\delta\omega$ ($\delta\omega_\mu$) is set at 2π•(200 Hz). The value of δ for this calculation is 1×10^{-4} Hz.

The Final Result

The final expression for $I(\omega)$ is obtained by combining equations (B-2), (B-4), and (B-6).

$$I(\omega)=\left(\frac{\omega_0}{\omega_r}\right)^2\int_{-\infty}^{\omega-\delta\omega}\frac{f(\omega;\omega_\beta)}{\tan\left(\frac{\pi\omega_r}{\omega_0}\frac{\omega-\omega_\beta}{\omega_\beta}\right)}d\omega_\beta$$
$$+\left(\frac{\omega_0}{\omega_r}\right)^2\int_{\omega+\delta\omega}^{\infty}\frac{f(\omega;\omega_\beta)}{\tan\left(\frac{\pi\omega_r}{\omega_0}\frac{\omega-\omega_\beta}{\omega_\beta}\right)}d\omega_\beta \qquad \text{(B-12)}$$
$$+\frac{1}{\pi}\left(\frac{\omega_0}{\omega_r}\right)^3 PV\int_{\omega-\delta\omega}^{\omega+\delta\omega}\frac{\omega_\beta f(\omega;\omega_\beta)-\omega f(\omega;\omega)}{\omega-\omega_\beta}d\omega_\beta$$

Figure B- 2 shows a typical evaluation of $I(\omega)$ for the band 4 vertical cooling system of the Debuncher.

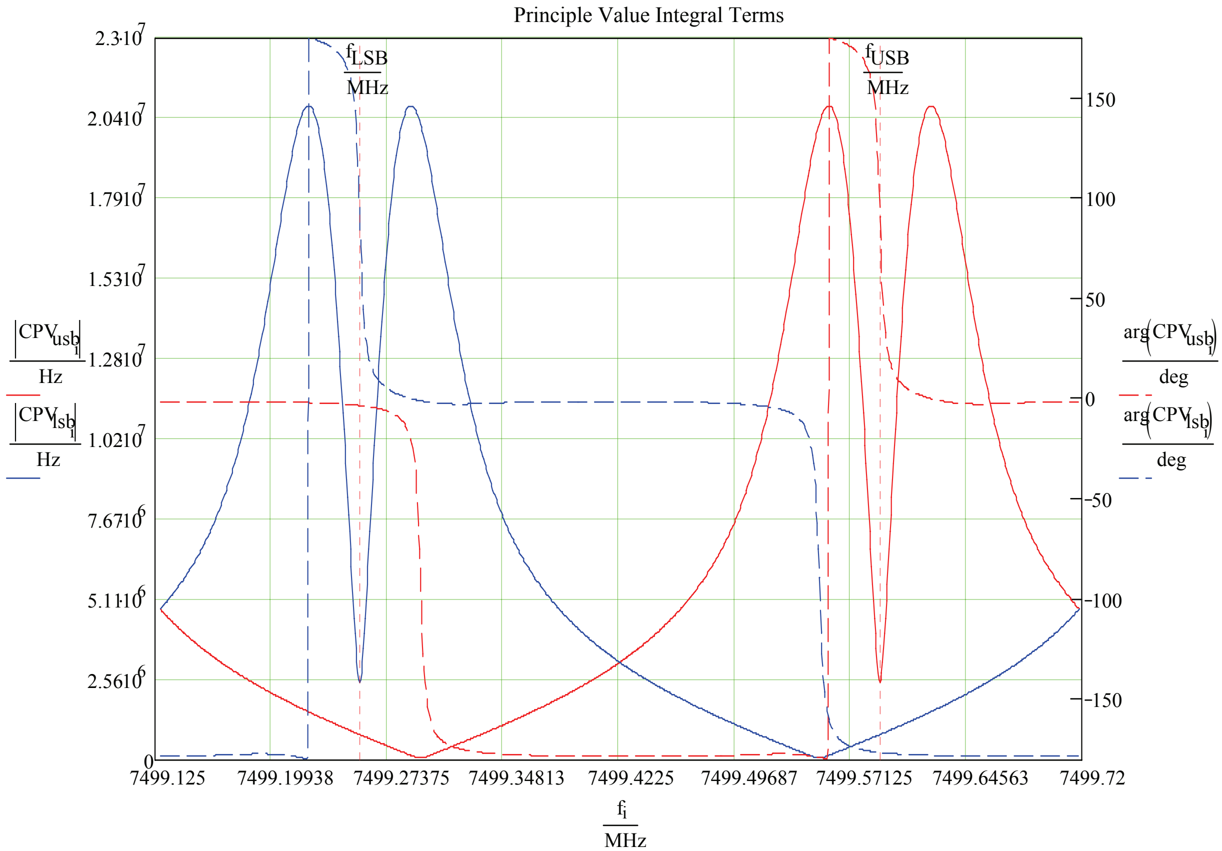

Figure B- 2 This graph shows the amplitude and phase of $I(\omega)$ for the upper sideband (CPV_{usb}) and the lower sideband (CPV_{lsb}). This plot comes from a calculation of sideband overlap for the Debuncher vertical band 4 cooling system.