Calculation of the Longitudinal Emittance of Unstacked Antiprotons from Wall Current Monitor Data

Pbar Note 698

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Introduction

The unstacking of stored antiprotons in the Accumulator has always involved significant RF manipulation of the beam. The same is true of antiproton transfers from the Recycler. Any dilution of the longitudinal emittance during the transfer of antiprotons will have adverse consequences latter on as beam is manipulated in the Main Injector for transfer into the Tevatron collider. Therefore it is important to monitor the longitudinal phase space occupied by the extracted beam.

In Collider Runs I and II the Antiproton Source department has used various wall monitor devices in the Accumulator or the AP1 beam line as a means of measuring the longitudinal emittance of unstacked antiprotons. The purpose of this note is to show how the signal obtained from these devices is used to determine the longitudinal emittance of the beam.

The Wall Current Monitor Signal

A wall monitor consists of a ceramic gap in the beam pipe that interrupts the image current of the beam that would otherwise be present there. For relativistic beam, the image current is exactly equal and opposite the beam current. At the gap the image current is redirected through the wall monitor electronics. The resulting signal is an accurate representation of beam current as a function of time. This signal measures the projection of the beam distribution on the time axis of longitudinal phase space.

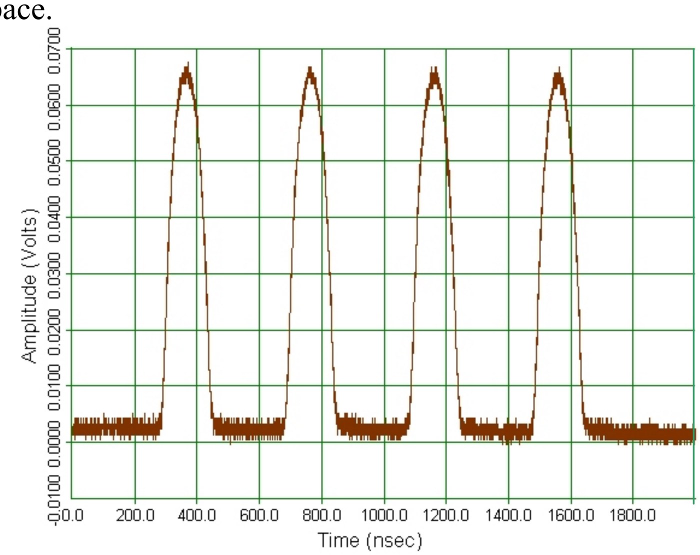

Figure 1 AP1 wall monitor signal for an antiproton transfer. The beam was bunched with a 2.5 MHz RF system prior to extraction from the Accumulator. The peak of each bunch corresponds to the center of a stationary ARF4 bucket.

The AP1 wall monitor beam signal is connected to a Tektronix TDS 3054 oscilloscope (500 MHz bandwidth, up to 5 GS/s sample rate) that is triggered by the passage of beam from the Accumulator. The 10,000-point oscilloscope trace is read out by an accelerator console application for analysis. The resulting data is shown in Figure 1.

A Hamiltonian for $\bar{p}$ Motion in Longitudinal Phase Space

The time evolution of energy and phase for antiproton beam in a stationary sinusoidal RF bucket is given by[1]:

$$\dot{\phi} = \frac{\omega_{RF}\eta}{\beta_0^2 E_0}\delta E \tag{1}$$

$$\dot{\delta E} = -eVf_0 \sin\phi \tag{2}$$

where the variables are defined as follows:

f_0 = revolution frequency of the synchronous particle
E_0 = synchronous energy
β_0 = synchronous velocity (as a fraction of c)
η = slip factor $\equiv \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)$
ω_{RF} = $2\pi h f_0$ = RF angular frequency (h = RF harmonic number)
V = total RF voltage amplitude (i.e. number of cavities × voltage/cavity)
δE = deviation from synchronous energy
ϕ = phase relative to synchronous particle $= \omega_{RF}(t - t_0)$

$\dot{\phi}$ and $\dot{\delta E}$ are the time derivatives of ϕ and δE respectively.

Equations (1) and (2) can be derived from the following Hamiltonian[2]:

$$H(\phi, \delta E) = \frac{1}{2}\alpha\left(\frac{\delta E}{\omega_{RF}}\right)^2 + U(\phi) \tag{3}$$

where

$$\alpha = \frac{\omega_{RF}^2\eta}{\beta_0^2 E_0} \tag{4}$$

[1] These equations are similar to those given in D.A. Edwards and M.J. Syphers *An Introduction to the Physics of High Energy Accelerators* in their chapter on Phase Stability and Acceleration. Equations 2.39 and 2.40 in Edwards and Syphers are identical to equations (1) and (2) above if eV is changed to $-eV$ in equation 2.40 to reflect the negative charge of an antiproton, and with the following additional (mostly notational) replacements:

$\frac{d}{dn} \to \frac{1}{f_0}\frac{d}{dt}$; $\tau \to \frac{1}{f_0}$; $\Delta E \to \delta E$; $E_s \to E_0$; $\frac{v^2}{c^2} \to \beta_0^2$; $\phi_s = 0$.

[2] The motivation for this form can be seen by differentiating equation (6) and observing that

$$\int \ddot{\phi}\dot{\phi}\,dt = \int \ddot{\phi}\,d\phi = \frac{1}{2}\int \frac{d}{dt}\left(\dot{\phi}^2\right)dt$$

$$\therefore \quad \frac{1}{2}\dot{\phi}^2 = \int \ddot{\phi}\,d\phi + const$$

and $U(\phi)$ is defined by:

$$U(\phi) = \frac{eV}{\pi h}\sin^2\frac{\phi}{2} \tag{5}$$

That this Hamiltonian will yield the correct equations of motion is seen by identifying $p_\phi \equiv \delta E/\omega_{RF}$ as the momentum that is conjugate to the coordinate variable ϕ. Hamilton's equations then give:

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \alpha\left(\frac{\delta E}{\omega_{RF}}\right) \tag{6}$$

$$\dot{p}_\phi = \frac{\delta\dot{E}}{\omega_{RF}} = -\frac{\partial H}{\partial \phi} = -\frac{eV}{2\pi h}\sin\phi \tag{7}$$

Equations (7) and (6) are identical to equations (1) and (2). It can therefore be concluded that the Hamiltonian of equation (3) properly represents the motion of an antiproton in an RF bucket and that the variables (ϕ, p_ϕ) are canonically conjugate.

Trajectories in Longitudinal Phase Space

The antiproton beam consists of particles whose trajectories in longitudinal phase space have a variety of amplitudes. The longitudinal emittance of the beam determined by calculating the phase space area enclosed by the trajectory of a particle whose amplitude is "representative[3]" of the beam. In this section expressions will be derived that give the phase space trajectory and the area enclosed by that trajectory[4].

Since $H(\phi, p_\phi)$ is a constant of the motion, the trajectories in longitudinal phase space can be calculated from:

$$H\left(\phi, p_\phi(\phi)\right) = H\left(\phi_m, 0\right) = U(\phi_m) = const \tag{8}$$

where ϕ_m is the phase at which an antiproton crosses the p_ϕ axis. The right and left hand sides of equation (8)are given by:

$$H\left(\phi_m, 0\right) = \frac{eV}{\pi h}\sin^2\frac{\phi_m}{2} \tag{9}$$

$$H\left(\phi, p_\phi\right) = \frac{1}{2}\alpha p_\phi^2 + \frac{eV}{\pi h}\sin^2\frac{\phi}{2} \tag{10}$$

The phase space trajectories are given by equating the right hand sides of equations (9) and (10). To avoid notation that confuses the phase space trajectory ($p_\phi(\phi)$ in equation (8)) with the canonical momentum, I will use the symbol W to denote the phase space trajectory function. The result is:

$$W\left(\phi, \phi_m\right) = \pm\sqrt{\frac{eV}{\pi h\alpha}\left(\cos\phi - \cos\phi_m\right)} \tag{11}$$

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[3] The meaning of "representative" depends on whether the emittance to be calculated is the emittance of the entire beam or some fraction of it. For example, to calculate the 95% emittance one must identify the phase space trajectory that encompasses 95% of the beam.

[4] A more detailed and sophisticated treatment of motion in longitudinal phase space can be found in G. Dôme, *Theory of RF Acceleration*, in the CERN Accelerator School Advanced Accelerator Physics course Proceedings, Queen's College, Oxford, England (Sept. 1985) CERN 87-03.

To simplify the notation somewhat, define $W_m \equiv W(0,\frac{\pi}{2})$. W_m is given by:

$$W_m \equiv \sqrt{\frac{eV}{\pi h \alpha}} \tag{12}$$

Equation (11) can now be written:

$$W(\phi,\phi_m) = \pm W_m \sqrt{\cos\phi - \cos\phi_m} \tag{13}$$

The phase space trajectories for antiprotons of varying amplitudes in a stationary ARF4 bucket are shown in Figure 2.

Integration of $W(\phi,\phi_m)$ over ϕ in equation (13) gives the total phase space area, $S(\phi_m)$, enclosed by the trajectory of a particle with amplitude ϕ_m.

$$S(\phi_m) = 4W_m \int_0^{\phi_m} \sqrt{\cos\phi - \cos\phi_m}\, d\phi \tag{14}$$

where advantage has been taken of the symmetry of the phase space trajectories about the p_ϕ and ϕ axes.

Since p_ϕ and ϕ are canonically conjugate variables, the phase space area, $S(\phi_m)$, is invariant under canonical transformations (i.e. the area calculated in ϕ, p_ϕ will be the same as the area calculated using any other pair of canonically conjugate variables). Therefore, area in the ϕ, p_ϕ plane is a meaningful quantity with which to characterize the beam[5].

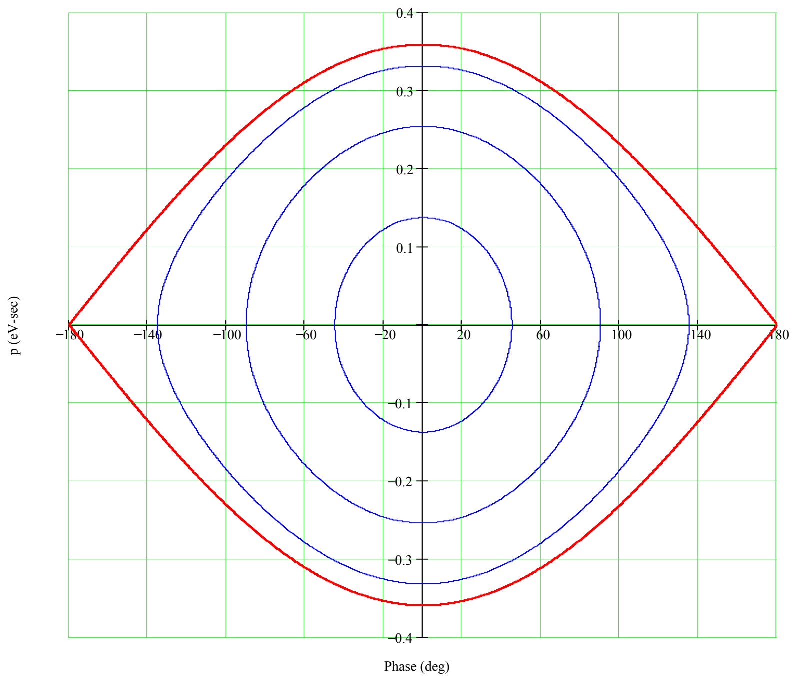

Figure 2 Antiproton phase space trajectories in a stationary ARF4 ($h = 4$) bucket. The cavity voltage amplitude is 550 V. The red curve is the separatrix. The blue curves are phase space trajectories for particles with several different amplitudes.

[5] $S(\phi_m)$ is also related to the action, J ($S(\phi_m) = 2\pi J$). Therefore S is an adiabatic invariant. If the parameters of H (e.g. RF voltage and synchronous phase) are varied adiabatically, the value of S will be preserved.

The Separatrix and Bucket Area

The separatrix is the phase space trajectory that constitutes the boundary between stable motion and unbounded motion. The separatrix crosses the p_ϕ-axis at the unstable fixed point[6] $\phi = \pi$. The equation for the separatrix, $W_s(\phi)$, is obtained by setting $\phi_m = \pi$ in equation (11):

$$W_s(\phi) = \pm W_m \sqrt{1+\cos\phi} = \pm\sqrt{2} W_m \cos\frac{\phi}{2} \tag{15}$$

Integration of $W_s(\phi)$ over ϕ in equation (15) gives the total phase space area, B_0, of a stationary RF bucket.

$$B_0 = 4\sqrt{2} W_m \int_0^\pi \cos\frac{\phi}{2} d\phi = 8\sqrt{2} W_m = \frac{16\beta_0}{\omega_{RF}} \sqrt{\frac{eVE_0}{2\pi h\eta}} \tag{16}$$

The Beam Distribution

Calculation of the longitudinal emittance requires knowledge of the value of ϕ_m for the antiproton phase space trajectory that contains the desired fraction, F, of the beam. This particular value of ϕ_m will hereinafter be called ϕ_F. Once ϕ_F is known, the longitudinal emittance is just $S(\phi_F)$ as given in equation (14). The determination of ϕ_F is simplified by the following observations:

(a) The antiproton beam is bunched in a stationary bucket prior to extraction from the Accumulator. Therefore the phase space trajectories of the bunched beam in each bucket are will be symmetric about the ϕ and p_ϕ axes.

(b) The beam distribution is stationary in time – that is, the antiproton bunch is not rotating or filamenting in phase space. This means that the beam is matched to the RF bucket – the boundary of the bunch in phase space is not changing with time and is determined by the trajectory of the highest amplitude particles.

(c) There is no beam outside of the RF bucket.

From (b) it follows that the beam distribution is only a function of the amplitude of the motion in longitudinal phase space. Equivalently, we could say that the number of particles at any point in longitudinal phase space is solely determined by the value of the Hamiltonian at that point. Thus the distribution, ψ, is a function of the Hamiltonian. Accordingly, we can write[7]:

$$\psi(\phi, p_\phi) d\phi dp_\phi = \psi_H\left(H(\phi, p_\phi)\right) d\phi dp_\phi \tag{17}$$

Given these assumptions, the following steps accomplish the determination of the longitudinal emittance of the beam:

1. Determine $\psi_H(H)$ from the projected distribution, $\psi_\phi(\phi)$, measured by the wall monitor.
2. Calculate the distribution of antiproton maximum phases, $\psi_m\left(\phi_m(H)\right)$ from $\psi_H(H)$.
3. Calculate the value of ϕ_F using $\psi_m(\phi_m)$ from:

$$F = \int_0^{\phi_F} \psi_m(\phi_m) d\phi_m \tag{18}$$

4. Calculate the beam longitudinal emittance from $\varepsilon_L = S(\phi_F)$.

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[6] A fixed point is defined by $\dot{p}_\phi = \dot{\phi} = 0$. According to equations (6) and (7) this occurs when $p_\phi = 0$ and ϕ is either 0 or $\pm\pi$. $\phi = 0$ corresponds to the stable fixed point while $\phi = \pm\pi$ correspond to unstable fixed points.

[7] Note that $\psi(\phi, p_\phi) d\phi dp_\phi \neq \psi_H(H) dH$. ψ_H has the same dimensions as ψ.

Step 1: Determination of $\psi_H(H)$

$\psi_\phi(\phi)$ is the projection of $\psi_H(H)$ on the ϕ-axis. The projected distribution is given by:

$$\begin{aligned}\psi_\phi(\phi) &= \int_{-\infty}^{\infty}\psi_H\left(H(\phi,p_\phi)\right)dp_\phi \\ &= 2\int_0^{\infty}\psi_H\left(H(\phi,p_\phi)\right)dp_\phi\end{aligned} \tag{19}$$

The limits of integration reflect the presumed symmetry of ψ_H in p_ϕ.

For a stationary bucket H is symmetric about the p_ϕ and ϕ axes. Thus $\psi_\phi(\phi)$ is symmetric about $\phi = 0$ and we need only focus attention on the quadrant in phase space: $\{(\phi, p_\phi)\text{: } \phi \geq 0, p_\phi \geq 0\}$. In this quadrant H is a one-to-one function of p_ϕ for any value of ϕ. This allows changing the variable of integration in equation (19) from p_ϕ to H:

$$\psi_\phi(\phi) = 2\int_{H(\phi,0)}^{\infty}\psi_H\left(H\right)\frac{\partial p_\phi}{\partial H}\,dH \tag{20}$$

Manipulation of equation (10) gives:

$$\begin{aligned}&H\left(\phi,p_\phi\right) = \frac{1}{2}\alpha p_\phi^2 + U(\phi) \\ &p_\phi\left(H,\phi\right) = \sqrt{\frac{2}{\alpha}\left[H - U(\phi)\right]} \\ &\frac{\partial p_\phi}{\partial H} = \frac{1}{\alpha p_\phi}\end{aligned} \tag{21}$$

Equation (20) becomes:

$$\begin{aligned}\psi_\phi(\phi) &= \frac{2}{\alpha}\int_{H(\phi,0)}^{\infty}\frac{\psi_H\left(H\right)}{p_\phi}dH \\ &= \sqrt{\frac{2}{\alpha}}\int_{U(\phi)}^{\infty}\frac{\psi_H\left(H\right)}{\sqrt{H-U(\phi)}}dH\end{aligned} \tag{22}$$

Equation (22) is an integral equation that must be solved for $\psi_H(H)$. Leo Michelotti has derived an integral transform solution to problems of this type[8]. The essence of Michelotti's method is that the integral of equation (22) can be written in the following form:

$$g(x) = \int_{x^2}^{\infty}\frac{f(u)}{\sqrt{u-x^2}}du \tag{23}$$

where $f(u)$ is related to $\psi_H(H)$. Extraction of $f(u)$ is then accomplished by the inverse relation:

$$uf(u) = \frac{1}{\pi}\int_{\infty}^{\sqrt{u}}\frac{\frac{d}{dx}\left(xg(x)\right)}{\sqrt{1-\frac{u}{x^2}}}dx \tag{24}$$

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[8] L. Michelotti, *Integral for longitudinal phase space tomography on equilibrium distributions*, Phys. Rev. Spec. Topics – Accel. and Beams, Vol. **6** 024001 (2003).

Following Michelotti's prescription, equation (22) can be cast into the form of equation (23) with the following substitutions:

$$\begin{aligned} x &\to \sqrt{U} \\ u &\to H \\ f(u) &\to \sqrt{\frac{2}{\alpha}}\psi_H(H) \\ g(x) &\to \psi_\phi\left(\phi\left(\sqrt{U}\right)\right) \end{aligned} \tag{25}$$

Since $U(\phi)$ is a positive, one-to-one function of ϕ in the upper right hand quadrant of phase space, the use of $\sqrt{U}$ as an independent variable is acceptable. Making these substitutions into equation (24) gives:

$$\sqrt{\frac{2}{\alpha}}H\psi_H\left(H\right) = \frac{1}{\pi}\int_{\infty}^{\sqrt{H}} \frac{\frac{d}{d\sqrt{U}}\left[\sqrt{U}\psi_\phi\left(\sqrt{U}\right)\right]}{\sqrt{1-H/U}} d\sqrt{U} \tag{26}$$

The lower integration limit of ∞ can be replaced by the value of $\sqrt{U}$ corresponding to the separatrix ($\sqrt{U(\pi)}$) since ψ_ϕ (and its derivative) are assumed to vanish outside of the RF bucket.

Equation (26) gives $\psi_H(H)$ in terms of the observed projected distribution $\psi_\phi(\phi)$.

Step 2: Determination of $\psi_m(\phi_m)$

Changing variables from $\sqrt{U}$ to ϕ, using equation (5), gives:

$$\sqrt{\frac{2}{\alpha}}H\psi_H\left(H\right) = \frac{eV}{\pi^2 h}\int_{\pi}^{\phi_m(H)} \frac{\frac{d}{d\phi}\left[\sin\frac{\phi}{2}\psi_\phi\left(\phi\right)\right]}{\sqrt{\frac{eV}{\pi h}\sin^2\frac{\phi}{2} - H}} \sin\frac{\phi}{2}\, d\phi \tag{27}$$

where $\phi_m(H)$ is given by:

$$\begin{aligned} H = H(\phi_m, 0) &= \frac{eV}{\pi h}\sin^2\frac{\phi_m}{2} \\ \phi_m(H) &= 2\sin^{-1}\left(\sqrt{\frac{\pi h}{eV}H}\right) \end{aligned} \tag{28}$$

The lower integration limit in equation (27) is the phase at which the separatrix crosses the ϕ axis in the right half-plane.

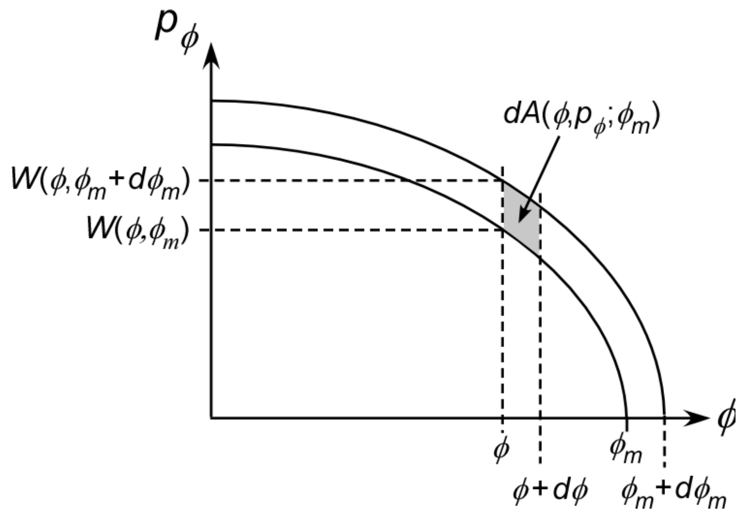

Figure 3 Two contours of constant H: $W(\phi,\ \phi_m)$ and $W(\phi,\ \phi_m + d\phi_m)$ for a stationary RF bucket. The differential phase space volume element $dA(\phi, p_\phi;\phi_m)$ in the integral of equation (29) is shown.

The distribution $\psi_H\left(H(\phi, p_\phi)\right)$ gives the particle density in a phase space volume element at the point (ϕ, p_ϕ). The ϕ_m distribution we need counts all the particles in a band in phase space between $W(\phi,\phi_m)$ and $W(\phi,\phi_m + d\phi_m)$. Thus the ϕ_m distribution, $\psi_m(\phi_m)$, is given by:

$$\psi_m(\phi_m)d\phi_m = \int_{\Gamma(\phi_m)} \psi_H\left(H\left(\phi, W(\phi,\phi_m)\right)\right) dA(\phi, p_\phi;\phi_m) \tag{29}$$

where the range of integration, $\Gamma(\phi_m)$, is the annular region in phase space between $W(\phi,\phi_m)$ and $W(\phi,\phi_m + d\phi_m)$. $dA(\phi, p_\phi;\phi_m)$ is the phase space volume element shown in Figure 3 and is given by:

$$dA(\phi, p_\phi;\phi_m) = dWd\phi = \left(\frac{\partial W(\phi,\phi_m)}{\partial \phi_m} d\phi_m\right) d\phi = \frac{1}{2} W_m \frac{\sin\phi_m}{\sqrt{\cos\phi - \cos\phi_m}} d\phi_m d\phi \tag{30}$$

Putting together equations (29) and (30) gives the following expression for $\psi_m(\phi_m)$:

$$\psi_m(\phi_m) = 2W_m \sin\phi_m \int_0^{\phi_m} \frac{\psi_H\left(H\left(\phi, W(\phi,\phi_m)\right)\right)}{\sqrt{\cos\phi - \cos\phi_m}} d\phi \tag{31}$$

Since the value of $H\left(\phi, W(\phi,\phi_m)\right) = U(\phi_m)$ is independent of ϕ, $\psi_H(H)$ can be factored out of the integral in equation (31). Thus,

$$\psi_m(\phi_m) = 2W_m \sin\phi_m \psi_H(H) \int_0^{\phi_m} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_m}} \tag{32}$$

In Appendix A it is shown that the integral in equation (32) is an elliptic integral:

$$\int_0^{\phi_m} \frac{d\phi}{\sqrt{\cos\phi - \cos\phi_m}} = \sqrt{2} F\left(\frac{\pi}{2} \middle| \sin\frac{\phi_m}{2}\right) \tag{33}$$

where F is an elliptic integral of the first kind[9].

[9] $F\left(\phi \middle| k\right) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}, \quad (k^2 < 1)$

$\psi_m(\phi_m)$ can now be written in terms of $\psi_H(H)$ as:

$$\psi_m(\phi_m) = 2\sqrt{2}W_m \sin\phi_m F\left(\frac{\pi}{2}\middle|\sin\frac{\phi_m}{2}\right)\psi_H\left(H(\phi_m)\right) \tag{34}$$

Finally, using equations (27) and (34) and writing H in terms of ϕ_m using equation (28) gives the sought after expression for $\psi_m(\phi_m)$:

$$\psi_m(\phi_m) = -\frac{4\sqrt{2}}{\pi}\frac{\sin\phi_m}{1-\cos\phi_m}F\left(\frac{\pi}{2}\middle|\sin\frac{\phi_m}{2}\right)\int_{\phi_m}^{\pi}\frac{\frac{d}{d\phi}\left[\sin\frac{\phi}{2}\psi_\phi(\phi)\right]}{\sqrt{\cos\phi_m-\cos\phi}}\sin\frac{\phi}{2}\,d\phi \tag{35}$$

There are two undesirable features of equation (35) that must be dealt with:

(1) The integral is singular at the lower limit of integration. A procedure for handling this singularity is given in Appendix C.

(2) The wall monitor signal, $\psi_\phi(\phi)$, must be differentiated. At the present time, this is handled by evaluating equation (35) for specific analytical forms for $\psi_\phi(\phi)$.

$\psi_m(\phi_m)$ was calculated for a parabolic and a Gaussian $\psi_\phi(\phi)$. The resulting $\psi_m(\phi_m)$ are shown in Figure 4 and Figure 5.

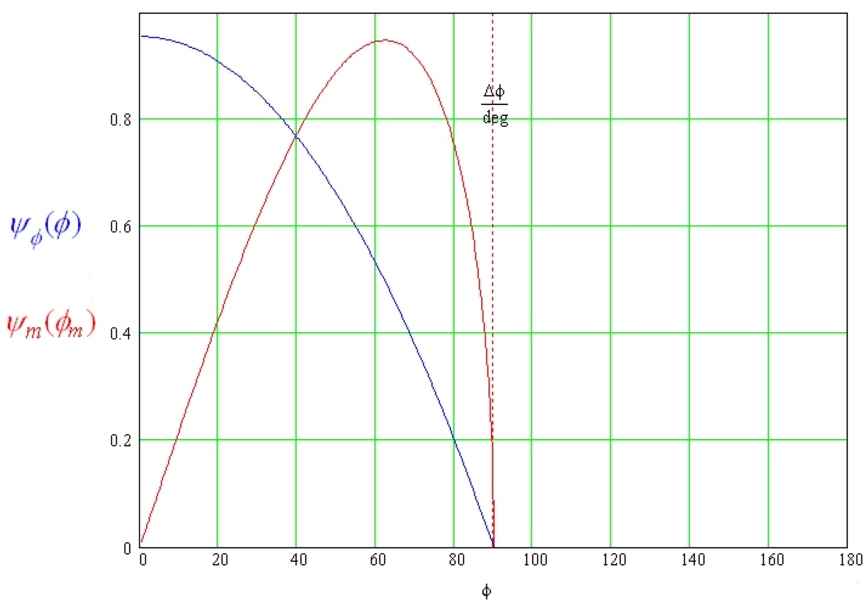

Figure 4 $\psi_\phi(\phi)$ (blue curve) and $\psi_m(\phi_m)$ (red curve) a parabolic beam distribution given by:

$$\psi_\phi(\phi) = \frac{3}{4\Delta\phi}\left[1-\left(\frac{\phi}{\Delta\phi}\right)^2\right] \quad \left(|\phi| \le \Delta\phi\right).\ \Delta\phi = 90° \text{ on this graph.}$$

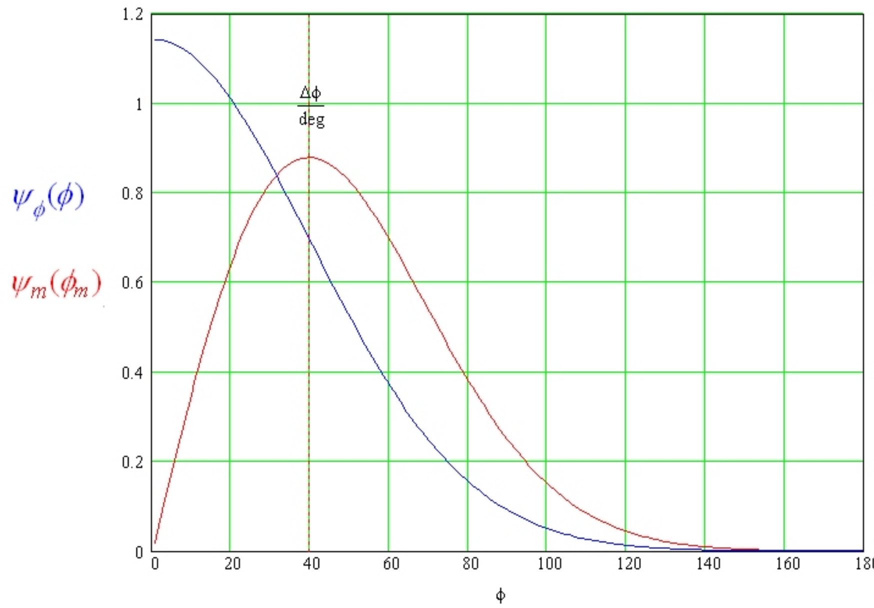

Figure 5 $\psi_\phi(\phi)$ (blue curve) and $\psi_m(\phi_m)$ (red curve) a Gaussian beam distribution where $\sigma = \Delta\phi = 40°$.

Step 3: Calculation of ϕ_F from $\psi_m(\phi_m)$

Rather than calculating $\psi_m(\phi_m)$ from the wall monitor data using equation (35) and then solving equation (18) for ϕ_F, it would be desirable to know a priori the value of $\mathcal{F}$ such that:

$$\mathcal{F} = \int_0^{\phi_F} \psi_\phi(\phi)\, d\phi \tag{36}$$

If $\mathcal{F}$ is known for a given value of F in equation (18), then ϕ_F, is determined directly from the wall monitor data and calculation of $\psi_m(\phi_m)$ is not required. This is the approach that has been taken for the longitudinal emittance calculation for beam from the Accumulator.

Experience has shown that the projected distribution of antiproton beam extracted from the Accumulator is well represented by a parabola (see Figure 6). The functional form of $\psi_\phi(\phi)$ used for this analysis is given by:

$$\psi_\phi(\phi) = \begin{cases} \dfrac{3}{4\Delta\phi}\left[1 - \left(\dfrac{\phi}{\Delta\phi}\right)^2\right], & |\phi| \le \Delta\phi \\ 0, & |\phi| > \Delta\phi \end{cases} \tag{37}$$

The value $\mathcal{F}$ for $F = 0.95$ in equation (18) was calculated for parabolic $\psi_\phi(\phi)$ of several widths. This was done by substitution of equation (37) into equation (35) to obtain $\psi_m(\phi_m)$ and then solving equation (18) for ϕ_F. The results are summarized in Table 1 below.

Table 1: The value of $\mathcal{F}$ for 95% of the beam for various $\Delta\phi$ for a parabolic ψ_ϕ

$\Delta\phi$	$\mathcal{F}$
20°	0.9927
40°	0.9927
60°	0.9925
80°	0.9924
100°	0.9921

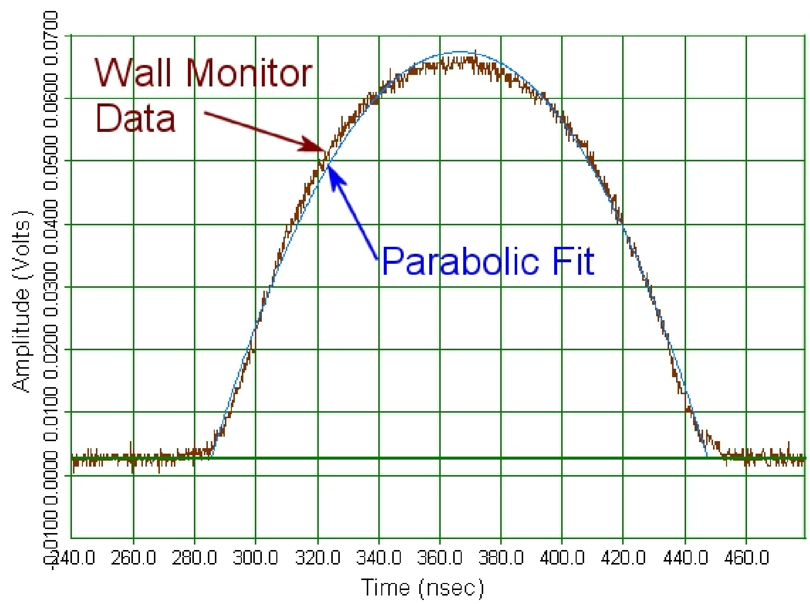

Figure 6 Parabolic fit to AP1 wall monitor data for a single 2.5 MHz bunch. The center of the bunch corresponds to the center of the RF bucket ($\phi = 0$). If t_0 is the time corresponding to the center of the bunch, the phase, ϕ, is related to time according to: $\phi = \omega_{RF}\left(t - t_0\right)$.

For a parabolic $\psi_\phi(\phi)$ the value of $\mathcal{F}$ varies by less than 0.1% over a wide range of widths. Thus, the calculation 95% longitudinal emittance for antiprotons transferred from the Accumulator is calculated by solving equation (36) for ϕ_F with $\mathcal{F} = 0.992$.

Step 4: Calculation of Longitudinal Emittance

Once ϕ_F is determined from the wall monitor data, the longitudinal emittance is calculated from $\varepsilon_L = S(\phi_F)$. From equation (14), the longitudinal emittance is given by:

$$\varepsilon_L = 4W_m \int_0^{\phi_F} \sqrt{\cos\phi - \cos\phi_F}\, d\phi \tag{38}$$

It is shown in Appendix B that the integral in equation (38) can be expressed in terms of elliptic integrals as:

$$\int_0^{\phi_F} \sqrt{\cos\phi - \cos\phi_F}\, d\phi = 2\sqrt{2}\left[E\left(\frac{\pi}{2}\middle|\sin\frac{\phi_F}{2}\right) - \cos^2\frac{\phi_F}{2} F\left(\frac{\pi}{2}\middle|\sin\frac{\phi_F}{2}\right)\right] \tag{39}$$

where E is an elliptic integral of the second kind[10]. Thus, the final expression for the longitudinal emittance becomes:

$$\begin{aligned}\varepsilon_L &= 8\sqrt{2}W_m\left[E\left(\frac{\pi}{2}\middle|\sin\frac{\phi_F}{2}\right) - \cos^2\frac{\phi_F}{2} F\left(\frac{\pi}{2}\middle|\sin\frac{\phi_F}{2}\right)\right] \\ &= B_0\left[E\left(\frac{\pi}{2}\middle|\sin\frac{\phi_F}{2}\right) - \cos^2\frac{\phi_F}{2} F\left(\frac{\pi}{2}\middle|\sin\frac{\phi_F}{2}\right)\right]\end{aligned} \tag{40}$$

Where B_0 is the bucket area given by equation (16).

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[10] $E(\phi\,|\,k) = \int_0^{\phi} \sqrt{1 - k^2\sin^2 x}\, dx \qquad (k^2 < 1)$

Appendix A

Evaluation of $I(\alpha)=\int_0^{\alpha}\frac{d\phi}{\sqrt{\cos\phi-\cos\alpha}}$

In this appendix it will be shown that

$$I(\alpha)=\int_0^{\alpha}\frac{d\phi}{\sqrt{\cos\phi-\cos\alpha}}=\sqrt{2}F\left(\frac{\pi}{2}\middle|\sin\frac{\alpha}{2}\right) \tag{1}$$

Where $F(\phi|K)$ is an elliptic integral of the first kind given by:

$$F\left(\phi|K\right)=\int_0^{\phi}\frac{dx}{\sqrt{1-K^2\sin^2 x}} \tag{2}$$

First change of variable:

Define a new variable θ such that:

$$\theta=\frac{\phi}{2} \qquad d\phi=2d\theta$$

$$\cos\phi=\cos 2\theta=1-2\sin^2\theta$$

With this change $I(\alpha)$ becomes:

$$I(\alpha)=2\int_0^{\frac{\alpha}{2}}\frac{d\theta}{\sqrt{1-\cos\alpha-2\sin^2\theta}} \tag{3}$$

Define a constant K:

$$K\equiv\sin\frac{\alpha}{2} \tag{4}$$

Note that $K^2\leq 1$. Then

$$1-\cos\alpha=2\sin^2\frac{\alpha}{2}=2K$$

$I(\alpha)$ becomes:

$$I(\alpha)=\frac{\sqrt{2}}{K}\int_0^{\frac{\alpha}{2}}\frac{d\theta}{\sqrt{1-\frac{\sin^2\theta}{K^2}}} \tag{5}$$

Second change of variable:

Define a new variable φ such that:

$$\sin\varphi = \frac{\sin\theta}{K} \tag{6}$$

with this substitution:

$$\begin{aligned}\cos\varphi\, d\varphi &= \frac{1}{K}\cos\theta\, d\theta \\ &= \frac{1}{K}\sqrt{1-\sin^2\theta}\, d\theta \\ &= \frac{1}{K}\sqrt{1-K^2\sin^2\varphi}\, d\theta \\ d\theta &= \frac{K\cos\varphi\, d\varphi}{\sqrt{1-K^2\sin^2\varphi}}\end{aligned}$$

Also, the integrand in equation (5) becomes:

$$\sqrt{1-\frac{\sin^2\theta}{K^2}} = \sqrt{1-\sin^2\varphi} = \cos\varphi$$

The new upper limit of integration is:

$$\varphi(\alpha) = \sin^{-1}\left(\frac{\sin\frac{\alpha}{2}}{K}\right) = \sin^{-1}(1) = \frac{\pi}{2}$$

With the variable change of equation (6), $I(\alpha)$ becomes:

$$I(\alpha) = \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-K^2\sin^2\varphi}} \tag{7}$$

Therefore:

$$\int_0^{\alpha} \frac{d\phi}{\sqrt{\cos\phi-\cos\alpha}} = \sqrt{2}F\left(\frac{\pi}{2}\,\middle|\,\sin\frac{\alpha}{2}\right) \tag{8}$$

Appendix B

Evaluation of $I(\alpha)=\int_0^{\alpha}\sqrt{\cos\phi-\cos\alpha}\,d\phi$

In this appendix it will be shown that:

$$\int_0^{\alpha}\sqrt{\cos\phi-\cos\alpha}\,d\phi=2\sqrt{2}\left[E\left(\frac{\pi}{2}\middle|\sin\frac{\alpha}{2}\right)-\cos^2\frac{\alpha}{2}F\left(\frac{\pi}{2}\middle|\sin\frac{\alpha}{2}\right)\right] \tag{1}$$

Where $F(\phi\,|K)$ is an elliptic integral of the first kind defined in equation (2) of Appendix A and $E(\phi\,|K)$ is an elliptic integral of the second kind defined by:

$$E\left(\phi\middle|K\right)=\int_0^{\phi}\sqrt{1-K^2\sin^2 x}\,dx \tag{2}$$

The procedure is identical to that of Appendix A.

First change of variable:

Let $\theta=\dfrac{\phi}{2}$. $I(\alpha)$ becomes:

$$I(\alpha)=2\sqrt{2}\sin\frac{\alpha}{2}\int_0^{\frac{\alpha}{2}}\sqrt{1-\frac{\sin^2\theta}{\sin^2\frac{\alpha}{2}}}\,d\theta \tag{3}$$

Define a constant K: $K\equiv\sin\dfrac{\alpha}{2}$. Equation (3) becomes:

$$I(\alpha)=\sqrt{2}K\int_0^{\frac{\alpha}{2}}\sqrt{1-\frac{1}{K^2}\sin^2\theta}\,d\theta \tag{4}$$

Second change of variable:

Define a new variable φ such that $\dfrac{1}{K}\sin\theta=\sin\varphi$. With this change $d\theta$ becomes:

$$d\theta=\frac{K\cos\varphi\,d\varphi}{\sqrt{1-K^2\sin^2\varphi}} \tag{5}$$

Also, the integrand in equation (4) becomes:

$$\sqrt{1-\frac{1}{K^2}\sin^2\theta}=\sqrt{1-\sin^2\varphi}=\cos\varphi \tag{6}$$

The limits of integration are changed to:

$$\varphi\left(\frac{\phi_m}{2}\right) = \sin^{-1}\left(\frac{1}{K}\sin\frac{\phi_m}{2}\right) = \sin^{-1}(1) = \frac{\pi}{2}$$
$$\varphi(0) = 0 \tag{7}$$

$I(\alpha)$ becomes:

$$I(\alpha) = 2\sqrt{2}K^2 \int_0^{\frac{\pi}{2}} \frac{\cos^2\varphi}{\sqrt{1-K^2\sin^2\varphi}}\,d\varphi \tag{8}$$

Re-write $\cos^2\varphi$ as follows:

$$\begin{aligned}\cos^2\varphi &= 1-\sin^2\varphi \\ &= 1-\frac{1}{K^2}+\frac{1}{K^2}\left(1-K^2\sin^2\varphi\right)\end{aligned}$$

$I(\alpha)$ becomes:

$$\begin{aligned}I(\alpha) &= 2\sqrt{2}K^2 \int_0^{\frac{\pi}{2}} \frac{1-\frac{1}{K^2}+\frac{1}{K^2}\left(1-K^2\sin^2\varphi\right)}{\sqrt{1-K^2\sin^2\varphi}}\,d\varphi \\ &= 2\sqrt{2}K^2\left\{\left(1-\frac{1}{K^2}\right)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-K^2\sin^2\varphi}}+\frac{1}{K^2}\int_0^{\frac{\pi}{2}}\sqrt{1-K^2\sin^2\varphi}\,d\varphi\right\} \\ &= 2\sqrt{2}\left[-\cos^2\frac{\alpha}{2}F\left(\frac{\pi}{2}\middle|\sin\frac{\alpha}{2}\right)+E\left(\frac{\pi}{2}\middle|\sin\frac{\alpha}{2}\right)\right]\end{aligned} \tag{9}$$

Appendix C

Handling the singularity of equation (35)

The procedure given here is essentially that given by Leo Michelotti in the paper already cited in footnote 8 on page 6.

To simplify the notation, re-write equation (35) as follows:

$$\psi_m\left(\phi_m\right) = G_0(\phi_m)\int_{\phi_m}^{\pi} \frac{G_1(\phi)}{\sqrt{\cos\phi_m - \cos\phi}}\, d\phi \tag{1}$$

where

$$\begin{aligned} G_0(\phi_m) &= -\frac{4\sqrt{2}}{\pi}\frac{\sin\phi_m}{1-\cos\phi_m} F\left(\frac{\pi}{2}\middle|\sin\frac{\phi_m}{2}\right) \\ G_1(\phi) &= \frac{d}{d\phi}\left[\sin\frac{\phi}{2}\psi_\phi\left(\phi\right)\right]\sin\frac{\phi}{2} \end{aligned} \tag{2}$$

The singularity at $\phi = \phi_m$ can be removed by rewriting equation (1) as follows:

$$\psi_m\left(\phi_m\right) = G_0(\phi_m)\left[\int_{\phi_m}^{\pi} \frac{G_1(\phi) - G_1(\phi_m)}{\sqrt{\cos\phi_m - \cos\phi}}\, d\phi + G_1(\phi_m)\int_{\phi_m}^{\pi} \frac{d\phi}{\sqrt{\cos\phi_m - \cos\phi}}\right] \tag{3}$$

The first integral is non-singular. The second integral in equation (3) can be written as an elliptic integral of the first kind as follows:

$$\int_{\phi_m}^{\pi} \frac{d\phi}{\sqrt{\cos\phi_m - \cos\phi}} = \int_0^{\pi-\phi_m} \frac{dx}{\sqrt{\cos x - \cos(\pi - \phi_m)}} = \sqrt{2}F\left(\frac{\pi}{2}\middle|\cos\frac{\phi_m}{2}\right) \tag{4}$$

Putting this back into equation (3) gives the final, manifestly non-singular, result[11]:

$$\psi_m\left(\phi_m\right) = G_0(\phi_m)\left[\int_{\phi_m}^{\pi} \frac{G_1(\phi) - G_1(\phi_m)}{\sqrt{\cos\phi_m - \cos\phi}}\, d\phi + \sqrt{2}G_1(\phi_m)F\left(\frac{\pi}{2}\middle|\cos\frac{\phi_m}{2}\right)\right] \tag{5}$$

[11] Note: care must be taken near $\phi_m = 0$. Several components of equation (5) blow up there. Equation (34) indicates that $\psi_m(0) = 0$.