Physics 105: Classical Mechanics, Sec. 2 (Strovink)
EXAM 1 Solutions
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Problem 1 (25 points): A *passive* transformation E consists, in the following order, of (I) a coordinate system reflection $(x_1, x_2, x_3) \to (-x_1, x_2, x_3)$, (II) A counterclockwise rotation of the coordinate system by $\pi/2$ about $\mathbf{e}_3$, and (III) A coordinate system reflection $(x_1, x_2, x_3) \to (x_1, x_2, -x_3)$.
(a) (15 points) If $\tilde{x}' = E\tilde{x}$, where $\tilde{x}'$ and $\tilde{x}$ are 3×1 matrix representations of the same vector in the transformed and untransformed coordinate systems, respectively, write the 3×3 matrix E.
(b) (10 points) Could the rotation represented by E be built up out of an infinite sequence of normal (not parity-inverting) infinitesimal rotations? Explain.

Solution: (a) The transformation E can be decomposed into the product $E_3E_2E_1$, where E_i is the ith transformation described in the problem statement. These three transformations are

$$\begin{aligned} E_1 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ E_2 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ E_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

Note that there is a relative minus sign from what you would expect for the upper left 2×2 block of E_2, since we are rotating passively in a *left handed* coordinate system due to the parity inversion of E_1. Multiplying these transformations together in the order indicated gives

$$E = E_3E_2E_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(b) To see whether this transformation can be represented by an infinite sequence of proper rotations, we take the determinant of the matrix representing E, giving $\det E = +1$. Since this matrix has positive determinant, it is a proper rotation, and any finite proper rotation can be built up from an infinite product of infinitesimal rotations. The reason for this is that the subgroup $SO(3)$ of $O(3)$ consisting of proper rotations is path-connected.

Problem 2 (35 points): During $-\infty < t < 0$, a linear oscillator satisfying the equation of motion

$$\ddot{x} + \omega_0 \dot{x} + \omega_0^2 x = \frac{F_x(t)}{m}$$

is driven at its resonant frequency by a for per unit mass

$$\frac{F_x(t)}{m} = a_0 \cos \omega_0 t,$$

where a_0 and ω_0 are positive real constants.
(a) (20 points) Find $x(0)$ and $\dot{x}(0)$ at $t = 0$.
(b) (15 points) At $t = 0$ the driving force is turned off. Find $x(t)$ for $t > 0$.

<u>Solution:</u> (a) Since the driving force was first applied long ago at $t = -\infty$, the effects of that initial transient have died out and can be ignored; for $t < 0$ all we need is a particular solution. To get it, as usual we substitute

$$x = \Re(A \exp(i\omega_0 t))$$

into the differential equation and choose to solve the complex version of the result, rather than its real part. Cancelling the common factor $\exp(i\omega_0 t)$, we obtain

$$\begin{aligned}
(-\omega_0^2 + i\omega_0^2 + \omega_0^2)A &= a_0 \\
A &= -\frac{ia_0}{\omega_0^2} \\
x(t < 0) &= \frac{a_0}{\omega_0^2} \sin \omega_0 t \\
x(0) &= 0 \\
\dot{x}(0) &= \frac{a_0}{\omega_0}.
\end{aligned}$$

(b) Here we need a solution $x_h(t)$ to the homogeneous equation. Substituting

$$x_h = \Re(B \exp(i\omega t))$$

with ω a constant to be determined, and cancelling the common factor $\exp(i\omega t)$, we obtain

$$\begin{aligned}
0 &= -\omega^2 + i\omega_0\omega + \omega_0^2 \\
\omega &= \frac{i\omega_0 \pm \sqrt{-\omega_0^2 + 4\omega_0}}{2} \\
&= -\frac{i\omega_0}{2} \pm \sqrt{\frac{3}{4}}\omega_0 \\
x_h(t) &= B \exp(-\frac{1}{2}\omega_0 t) \cos(\sqrt{\frac{3}{4}}\omega_0 t + \beta),
\end{aligned}$$

where B and β are adjustable constants. (This standard underdamped solutions may also simply be recalled from memory or from notes.) Enforcing the initial condition $x(0) = 0$, we take $\beta = \frac{\pi}{2}$ and the solution becomes

$$x(t) = -B \exp(-\frac{1}{2}\omega_0 t) \sin(\sqrt{\frac{3}{4}}\omega_0 t).$$

Matching the remaining initial condition,

$$\begin{aligned}
\frac{a_0}{\omega_0} &= \dot{x}(0) \\
&= -B\sqrt{\frac{3}{4}}\omega_0 \\
\sqrt{\frac{4}{3}}\frac{a_0}{\omega_0^2} &= -B \\
x(t > 0) &= \sqrt{\frac{4}{3}}\frac{a_0}{\omega_0^2} \exp(-\frac{1}{2}\omega_0 t) \sin(\sqrt{\frac{3}{4}}\omega_0 t).
\end{aligned}$$

Problem 3 (40 points): On a horizontal frictionless table, positions may be specified by Cartesian coordinates (x, y) or circular coordinates (ρ, θ), where as usual $\rho \equiv \sqrt{x^2 + y^2}$ and $\theta \equiv \arctan y/x$. The table supports a very small puck of mass m; at the origin, a small hole is drilled through the table. By means of a frictionless massless string that is threaded through the hole, the puck is attached to a massless Hooke's law spring underneath the table. The lenght of the string is such that the spring is relaxed when the puck is at the origin. Therefore, away from the origin, the puck has potential energy

$$U = \frac{1}{2}k\rho^2,$$

where k is the spring constant.
(a) (10 points) Write down the Lagrangian *in circular coordinates*
(b) (10 points) Find two constants of the puck's motion. Exapress them in terms of $\rho, \dot{\rho}, \theta, \dot{\theta}$, and constants.
(c) (10 points) Use the Euler-Lagrange equation to obtain the second-order differential equation of motion for the puck's radius ρ. Substitute a constant of the motion to eliminate $\dot{\theta}$ from this equation.
(d) (10 points) Under the initial conditions $x(0) = x_0 \neq 0$, $\dot{x}(0) = \dot{y}(0) = y(0) = 0$, solve for the motion of the puck.

Solution: (a) The Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2) - \frac{1}{2}k\rho^2.$$

(b) Since $\mathcal{L}$ is cyclic in (does not depend on) the coordinate θ, we know that

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \ell_\theta$$

will be a constant of the motion. In addition, since $\mathcal{L}$ does not depend explicitly on time, we have

$$-\frac{\partial \mathcal{L}}{\partial t} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = 0,$$

so the Hamiltonian will also be a constant of the motion. These two constants are

$$\begin{aligned} \ell_\theta &= m\rho^2\dot{\theta} \\ \mathcal{H} &= \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\theta}^2) + \frac{1}{2}k\rho^2. \end{aligned}$$

(c) The equation of motion for ρ is

$$\frac{\partial \mathcal{L}}{\partial \rho} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\rho}} = m\rho\dot{\theta}^2 - k\rho - m\ddot{\rho} = 0.$$

Substituting $\dot{\theta} = \ell_\theta/m\rho^2$, we get

$$\ddot{\rho} + \frac{k}{m}\rho - \frac{\ell_\theta^2}{m^2\rho^3} = 0.$$

(d) The initial conditions imply that $\dot{\theta}(0) = 0$, so $\ell_\theta = 0$ for all time, and they also imply that the motion will occur entirely on the x axis. This means that we can write our differential equation as

$$\ddot{x} + \frac{k}{m}x = 0,$$

which has the solution

$$x(t) = x_0 \cos\sqrt{\frac{k}{m}}t$$

to give $x(0) = x_0$ and $\dot{x}(0) = 0$.