TENSOR or COMPONENT NOTATION for VECTORS

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First, a couple of reminders. In type set text, a vector is shown in bold face type *e.g.* **A**. In hand written text, a vector is shown by an arrow *e.g.* $\vec{A}$. The usual expansion of a vector in Cartesian coordinates is:

$$\mathbf{A} = \vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \qquad \text{where: } \hat{x},\hat{y},\hat{z} \text{ are the unit vectors in the x, y and z directions.} \qquad (1)$$

This notation for a vector is very compact and quite useful. However, there are occasions when a more direct way to do algebra or calculus on the components of a vector is required. The notation that allows this is called tensor or component notation. Both notations are very useful. You should become conversant with both. We describe tensor notation below.

Another Way to Represent a Vector

Let: A_1 stand for A_x and $\hat{x}_1$ stand for $\hat{x}$
A_2 stand for A_y and $\hat{x}_2$ stand for $\hat{y}$
A_3 stand for A_z and $\hat{x}_3$ stand for $\hat{z}$

Then (1) may be written as: $$\mathbf{A} = \sum_{i=1}^{3} A_i\,\hat{x}_i$$

Some short hand: $$\mathbf{A} = \sum_{i=1}^{3} A_i\,\hat{x}_i = A_i\,\hat{x}_i$$

This is the Einstein convention. Any index that is repeated, *i* in this case, we sum over the values 1 to 3.

More short hand:

We don't need $\hat{x}_i$! That is, A_i means the *i* th component of **A** or $\vec{A}$, the $\hat{x}_i$ is redundant. This is the crux of tensor notation. That is, just write A_i with $i = 1,2,3$ to stand for the any one of the components of **A**. (The extension of this concept to other than three dimensions is obvious.)

Examples of Use in Some Fundamental Vector Operations

Addition and subtraction:

$A_i = B_i \pm C_i$ is the *i* th component of $\mathbf{A} = \mathbf{B} \pm \mathbf{C}$.

Scalar or dot product:

$A_j B_j$ is the same as $\mathbf{A \cdot B}$ (remember, we sum over repeated indices).

Differential vector operators:

$\frac{\partial}{\partial x_i}$ is the i th component of $\boldsymbol{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

for example:

$\frac{\partial \phi}{\partial x_i}$ is the i th component of grad ϕ or $\boldsymbol{\nabla}\phi$

$\frac{\partial A_j}{\partial x_j}$ is the same as div $\mathbf{A}$ or $\boldsymbol{\nabla}\cdot\mathbf{A}$.

What about a cross product?

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}(A_y B_z - A_z B_y) + \hat{y}(A_z B_x - A_x B_z) + \hat{z}(A_x B_y - A_y B_x). \tag{2}$$

To see how to handle this in component or tensor notation, we need to digress into symbolic ways of writing and manipulating determinants.

<u>Digression into Determinants</u>

<u>First</u>, we define the Kronecker symbol δ_{ij}

$$\delta_{ij} \begin{cases} = 0 & \text{if } i \neq j \\ = 1 & \text{if } i = j \end{cases} \tag{3}$$

<u>Second</u>, we define the Levi Civita symbol ε_{ijk}

$$\varepsilon_{ijk} \begin{cases} = 0 & \text{if any two of } i, j, k \text{ are equal} \\ = +1 & \text{if } ijk = 123, 231 \text{ or } 312 \\ = -1 & \text{if } ijk = 321, 213 \text{ or } 132 \end{cases} \tag{4}$$

The values of *ijk* that give +1 are called even permutations, while those that give −1 are called odd permutations. This is because the number of interchanges of pairs of indices that are necessary to create the sequence 1,2,3 are even for the +1 case and odd for −1. Try it! A short hand for this definition is often found in physics texts. It goes something like "ε_{ijk} is the totally antisymetric quantity with $\varepsilon_{123} = 1$".

Third, we look at the determinant of a 3×3 matrix *e.g.*

$$(a) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinate of the matrix (a) may be written as:

$$\det a = \varepsilon_{ijk}\, a_{1i}\, a_{2j}\, a_{3k} \text{ or } \det a = \varepsilon_{ijk}\, a_{i1}\, a_{j2}\, a_{k3} \tag{5}$$

Check it yourself. You will see that these are just the usual row or column expansions. The ε_{ijk} just takes care of the alternating signs.

Now Back to the Cross Product:

In light of (2) and (5), we see that the i th component of $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is:

$$C_i = \varepsilon_{ijk}\, A_j B_k$$

To make use of this, we need to learn more of the algebra of δ_{ij} and ε_{ijk}.

Algebra of δ_{ij}:

$$\delta_{ij}\, \delta_{jk} = \delta_{ik} \tag{6}$$

$$\delta_{kk} = 3 \tag{7}$$

To prove these, just write out the indicated sum.

Algebra of ε_{ijk}:

$$\varepsilon_{ijk}\, a_{il} a_{jm} a_{kn} = \varepsilon_{lmn} \det a. \tag{8}$$

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{bmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{bmatrix} \tag{9}$$

The proof of each of these is similar. Below, I prove (9) and leave the proof of (8) as an exercise.

1. If any two of i,j,k or any two of k,l,m are equal, the left side = 0 by the definition of ε_{ijk} (4), and the right side is 0 as the determinant has either two equal rows or two equal columns.

2. For i,j,k = 123 and l,m,n = 123, the expression is clearly true. The left side is 1×1 and the right side is the determinant of the unit matrix.

3. Under the interchange of any two of either i,j,k or l,m,n, the left side changes sign. The right side changes sign as this is the interchange of either two rows or two columns of the determinant which changes its sign.

Thus, all possible values of the indices give an equality.

Using (9), we can evaluate some very useful expressions.

$$\varepsilon_{ijk}\varepsilon_{kmn} = \begin{vmatrix} \delta_{ik} & \delta_{im} & \delta_{in} \\ \delta_{jk} & \delta_{jm} & \delta_{jn} \\ \delta_{kk} & \delta_{km} & \delta_{kn} \end{vmatrix} \quad \textit{i.e.}\text{ sum over one index } (k)$$

$$= \delta_{ik}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jk}\delta_{kn} - \delta_{jn}\delta_{kk}) + \delta_{in}(\delta_{jk}\delta_{km} - \delta_{jm}\delta_{kk})$$

by (5) and (6):

$$= \delta_{jm}\delta_{in} - \delta_{jn}\delta_{im} - \delta_{im}\delta_{jn} + 3\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} - 3\delta_{in}\delta_{jm}$$

or $\quad \varepsilon_{ijk}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$ $\quad$ Sum over one index $\quad$ (10)

and $\quad \varepsilon_{ijk}\varepsilon_{jkn} = -(\delta_{ij}\delta_{jn} - \delta_{in}\delta_{jj}) = 2\delta_{in}$ $\quad$ Sum over two indices $\quad$ (11)

and $\quad \varepsilon_{ijk}\varepsilon_{ijk} = 6$ $\quad$ Sum over all 3 indices $\quad$ (12)

Examples of Applications to Vectors

Example 1

You learned somewhere that:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \qquad \text{(Bac – Cab rule)}$$

We prove this using tensor notation.

$$[\mathbf{A} \times (\mathbf{B} \times \mathbf{C})]_i = \varepsilon_{ijk}A_j\varepsilon_{klm}B_lC_m = \varepsilon_{ijk}\varepsilon_{klm}A_jB_lC_m$$

by (10): $= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_jB_lC_m = B_i(A_jC_j) - C_i(A_jB_j)$

$= B_i(\mathbf{A}\cdot\mathbf{C}) - C_i(\mathbf{A}\cdot\mathbf{B})$

Example 2

$$[\nabla \times (\mathbf{A} \times \mathbf{B})]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\ \varepsilon_{klm}A_lB_m = \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial}{\partial x_j}\ A_lB_m$$

by (9): $= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\dfrac{\partial}{\partial x_j}\ A_lB_m$

$$= \frac{\partial}{\partial x_j}(A_iB_j) \quad - \frac{\partial}{\partial x_j}(A_jB_i) \quad = A_i\frac{\partial}{\partial x_j}\ B_j + B_j\frac{\partial}{\partial x_j}\ A_i - A_j\frac{\partial}{\partial x_j}\ B_i - B_i\frac{\partial}{\partial x_j}\ A_j$$

$$= A_i(\nabla\cdot\mathbf{B}) + (\mathbf{B}\cdot\nabla)A_i - (\mathbf{A}\cdot\nabla)B_i - B_i(\nabla\cdot\mathbf{A})$$

Example 3

The vector $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$ appears often.

We may write $\mathbf{r}$ in component notation as: $r_i = x_i$

Then:

$\text{div}\ \mathbf{r} = \dfrac{\partial}{\partial x_i}\ x_i\ = \delta_{ii}$ by (7) this is equal to 3

$(\text{curl}\ \mathbf{r})_i = \varepsilon_{ijk}\dfrac{\partial}{\partial x_j}\ x_k = \varepsilon_{ijk}\ \delta_{jk}$ by (3) and (4) is equal to 0

Other examples will be found on a future problem set.

<u>Determinants of Matrices of Other Orders</u>
(This section is for interest only. We will not need it for 110A)

Certainly determinants of matrices of order other than 3×3 are of interest and important. Our definition of the Levi Civita symbol was for 3×3 matrix. An examination of the definition for this case (4) shows that it can be defined in the same way for any size matrix. That is for an $n \times n$ matrix we just write $\varepsilon_{ijk....}$ with the dots indicating there are a total of n indices. It is the totally antisymetric quantity with $\varepsilon_{123...n} = 1$.

With this more general definition the equivalent of (5) becomes:

$$\det a = \varepsilon_{ijk\ldots} a_{1i}\, a_{2j}\, a_{3k} \ldots \tag{13}$$

For example for n = 2. we have:

$$\det a = \varepsilon_{ij}\, a_{1i} a_{2j}$$

or for n = 4:

$$\det a = \varepsilon_{ijkl}\, a_{1i} a_{2j} a_{3k} a_{4l}$$

The very important relations (8) and (9) generalize in the obvious way.

$$\varepsilon_{ijk\ldots\ldots}\, a_{il} a_{jm} a_{kn} \ldots\ldots = \varepsilon_{lmn\ldots\ldots} \det a. \tag{8'}$$

$$\varepsilon_{ijk\ldots\ldots} \varepsilon_{abc\ldots\ldots} = \begin{vmatrix} \delta_{ia} & \delta_{ib} & \delta_{ic} & \bullet & \bullet \\ \delta_{ja} & \delta_{jb} & \delta_{jc} & \bullet & \bullet \\ \delta_{ka} & \delta_{kb} & \delta_{kc} & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{vmatrix} \tag{9'}$$

If we contract two ε s on the last m indices it can be shown (I hate those words but they save space) to be:

$$\varepsilon_{\ldots\ldots}\varepsilon_{\ldots\ldots} \text{ contracted on the last m indices} = m! \times \det a' \tag{14}$$

where: a' is the first n−m minor of the matrix:

$$\begin{vmatrix} \delta_{ia} & \delta_{ib} & \delta_{ic} & \bullet & \bullet \\ \delta_{ja} & \delta_{jb} & \delta_{jc} & \bullet & \bullet \\ \delta_{ka} & \delta_{kb} & \delta_{kc} & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{vmatrix}$$

For example:

for n = 3 m = 1 the contraction is $\varepsilon_{ijk}\varepsilon_{abk} = 1! \times \begin{vmatrix} \delta_{ia} & \delta_{ib} \\ \delta_{ja} & \delta_{jb} \end{vmatrix}$

which is equivalent to (10).

or for n = 4 m = 2.

$$\varepsilon_{ijkl}\varepsilon_{abkl} = 2! \times (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja})$$

which is equivalent to (10) for a 4 × 4 matrix.

The cofactor of the ij th component of a matrix is defined as $(-1)^{i+j}$ times the determinant of the matrix with the i th row and the j th column removed. For an n $\times$ n matrix it can be formally written as:

$$c_{ij} = \frac{1}{(n-1)!} \quad \varepsilon_{iabc\ldots}\varepsilon_{j\alpha\beta\gamma\ldots} \, a_{a\alpha} \, a_{b\beta} \, a_{c\gamma} \ldots \tag{15}$$

Mathematicians call the transpose of the matrix made up of the elements c_{ij} the adjoint of the matrix a_{ij}. Written Adj A. **Note: this is not what we call adjoint in physics.** Using the algebra of the Levi Civita symbol it is easy to show:

$$A \text{ Adj } A = (1) \det A$$

$$\text{or} \qquad a_{ij} \, c_{jk}{}^{T} = a_{ij} \, c_{kj} = \delta_{ik} \det a \tag{16}$$

This leads to an expression for the inverse of a_{ij}:

$$A^{-1} = \frac{\text{Adj }(A)}{|A|}$$

$$\text{or} \qquad a_{ij}{}^{-1} = \frac{c_{ij}{}^{T}}{|\, a_{ij} \,|} \tag{17}$$

Try it with something simple like:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Now it is interesting to contemplate what all this means with respect to vectors. Look at the cross product in 3 dimensions. It was defined as:

$$C_i = \varepsilon_{ijk} \, A_j B_k.$$

Consider some other dimension. Can you make a similar definition? For example:

for $n = 2$ $\quad \varepsilon_{ij} \, A_i B_j$ $\quad$ or $\quad$ for $n = 4$ $\quad \varepsilon_{ijkl} \, A_k B_l$

The problem is these do not produce vectors. The first is a scalar (*i.e.* has only one value) and the second is a 4 $\times$ 4 matrix! The cross product, as we know it, exists only in 3 dimensions. We will learn much more about this as we learn about tensors and there meaning in physics.