Physics H7C — University of California, Berkeley — F'99 (Strovink)

RELATIVITY NOTES

1. SPECIAL RELATIVITY

1.1 SPACETIME

Because c = speed of light in vacuum is the same in all reference frames according to Maxwell's equations, we can imagine considering

$$ct = (m/sec)(sec) = (m)$$

to be the 0^{th} dimension in spacetime.

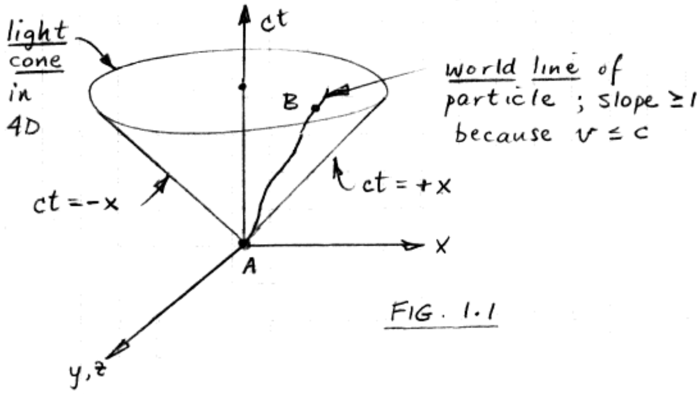


FIG. 1.1

An event is described by $r = (\overset{0}{ct}, \overset{1}{x}, \overset{2}{y}, \overset{3}{z})$. Because information travels at $\le c$, if event B is causally connected to event A, at the origin, event B must be within the light cone.

1.2 DISTANCE IN SPACETIME

What is r^2? Consider 2 non-accelerating ("inertial") reference frames

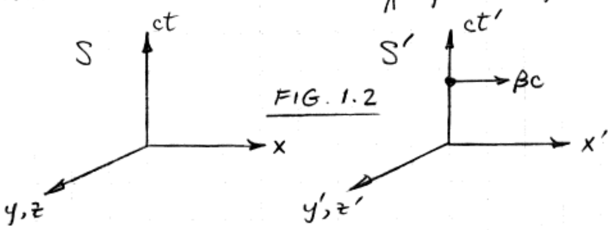


FIG. 1.2

We choose the origins to be the same, i.e. $x = y = z = 0$ is the same point as $x' = y' = z' = 0$ when $ct = ct' = 0$. Frame S′ is moving in the ($x = x'$) direction with respect to S with velocity βc.

A pulse of EM radiation is emitted at $(ct, x, y, z) = (ct', x', y', z') = 0$. In either frame it is a bubble expanding from the 3D origin:

$$x^2 + y^2 + z^2 = c^2t^2$$
$$x'^2 + y'^2 + z'^2 = c^2t'^2 .$$

$$\therefore\ c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 \qquad (1.1)$$

for the bubble.

So we define the distance Δr between 2 events r_A and r_B to be

$$\Delta r^2 = c^2(t_B - t_A)^2 \ominus (x_B - x_A)^2 \ominus (y_B - y_A)^2 \ominus (z_B - z_A)^2 \qquad (1.2)$$

Had we used $\oplus$ instead of $\ominus$, (1.1) would have forced the distance between 2 events to be different when viewed in different frames.

Distances between events are called
- timelike if $\Delta r^2 > 0$ $(c^2\Delta t^2 > |\Delta\vec{r}|^2)$
- lightlike if $\Delta r^2 = 0$ $(c^2\Delta t^2 = |\Delta\vec{r}|^2)$
- spacelike if $\Delta r^2 < 0$ $(c^2\Delta t^2 < |\Delta\vec{r}|^2)$.

Except for quantum mechanical effects, pairs of events can be causally connected only if the interval between them is timelike (within the light cone) or lightlike (on the light cone).

1.3 ROTATION IN 2D SPACE

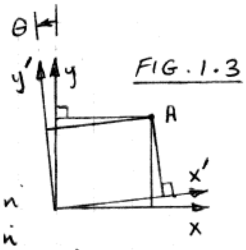


FIG. 1.3

$r = (x, y)$ and $r' = (x', y')$ are the coordinates of point A as viewed in S or S′. From the diagram, when $\theta \ll 1$ we obtain the infinitesimal transformation

$$x' = x + \theta y \qquad\qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$y' = -\theta x + y \qquad \text{or} \qquad\qquad\qquad (1.3)$$

The distance between point A and the origin is

$$r^2 = x^2 + y^2$$
$$r'^2 = x'^2 + y'^2 = (x + \theta y)^2 + (y - \theta x)^2 \quad \text{neglect}$$
$$= x^2 + \cancel{2\theta xy} + y^2 - \cancel{2\theta xy} + \cancel{\mathcal{O}(\theta^2)}$$
$$= r^2 \ \checkmark$$

When θ is not $\ll 1$, (1.3) becomes

$$\begin{aligned} x' &= \cos\theta\, x + \sin\theta\, y \\ y' &= -\sin\theta\, x + \cos\theta\, y \end{aligned} \quad \text{or} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad (1.4)$$

For non infinitesimal rotations it is still true that $r'^2 = r^2$ because $\sin^2\theta + \cos^2\theta = 1$.

1.4 INFINITESIMAL TRANSFORMATION IN 2D SPACETIME

$r = (ct, x)$ and $r' = (ct', x')$ are the coordinates of point A as viewed in S or S'. From the diagram, when $\beta \ll 1$,

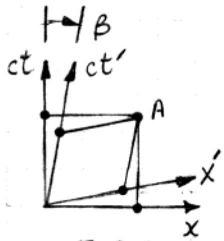


FIG 1.4

$$\begin{aligned} x' &= x \ominus \beta ct \\ ct' &= -\beta x + ct \end{aligned} \quad \text{or} \quad \begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \qquad (1.5)$$

Why did we draw the diagram in this peculiar way, requiring a $\ominus$ rather than the usual $\oplus$ sign where indicated? The distance between event A and the origin is

$$\begin{aligned} r^2 &= c^2t^2 - x^2 \\ r'^2 &= c^2t'^2 - x'^2 = (ct - \beta x)^2 - (x - \beta ct)^2 \\ &= (ct)^2 - \cancel{2\beta ctx} - x^2 \cancel{+ 2\beta ctx} + \underbrace{O(\beta^2)}_{\text{neglect}} \\ &= r^2 \ \checkmark \end{aligned}$$

The peculiar diagram is <u>necessary to force $r'^2 = r^2$</u>.

The nonrelativistic <u>Galilei transformation</u> is obtained from Eq. (1.5) by ignoring $-\beta x$ with respect to ct:

$$\begin{aligned} x' &= x - \beta ct = x - vt \\ t' &\approx t \end{aligned} \qquad (1.6)$$

You used this transformation (perhaps without realizing it) to solve distance = rate × time problems in high school.

The form of Eq. (1.5) which <u>exactly preserves distances in spacetime</u> is

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \frac{1}{\sqrt{1-\beta^2}} \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \qquad (1.7)$$

but this discussion so far pertains only to infinitesimal transformations.

1.5. FINITE TRANSFORMATION IN 2D SPACETIME

When β in Fig. 1.4 is not $\ll 1$, we call $\underbrace{\eta}_{\text{the rotation parameter}}$ ("eta") rather than β. η is called the "rapidity" or "boost" and, in general, is a function of β.

When the spacetime transformation is no longer infinitesimal, Eq. (1.5) becomes

$$\begin{aligned} x' &= (\cosh\eta)x - (\sinh\eta)ct \\ ct' &= -(\sinh\eta)x + (\cosh\eta)ct \end{aligned}$$

or $\qquad (1.8)$

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$

using the <u>hyperbolic functions</u>

$$\cosh a \equiv \frac{e^a + e^{-a}}{2} \qquad \sinh a \equiv \frac{e^a - e^{-a}}{2}$$

$$\tanh a \equiv \sinh a / \cosh a$$

$$\sinh(0) = 0, \ \cosh(0) = 1, \ \tanh(0) = 0$$

$$\sinh(\infty) = \infty, \ \cosh(\infty) = \infty, \ \tanh(\infty) = 1$$

$$\cosh^2 a - \sinh^2 a = 1$$

It is the <u>last</u> property which guarantees $r'^2 = r^2$ for finite transformations in spacetime.

Rewrite Eq. (1.8) as

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \cosh\eta \begin{bmatrix} 1 & -\tanh\eta \\ -\tanh\eta & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \qquad (1.9)$$

and compare to Eq. (1.7) making use of

$$\cosh^2\eta = \frac{\cosh^2\eta}{\cosh^2\eta - \sinh^2\eta} = \frac{1}{1 - \tanh^2\eta}.$$

Eqs. (1.7) and (1.9) are consistent if

$$\beta = \tanh\eta \ (\le 1), \quad \eta = \tanh^{-1}\beta \qquad (1.10)$$

$\Rightarrow \exists$ <u>no</u> faster-than-light particles (tachyons).

Defining $\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}$,

Eq. (1.9) becomes the <u>Lorentz transformation</u>

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \quad \text{or} \qquad (1.11)$$

$$\begin{aligned} x' &= \gamma x - \gamma\beta ct \\ ct' &= -\gamma\beta x + \gamma ct \end{aligned}.$$

1.6. GENERALIZATIONS OF LORENTZ TRANSFORMATION

- 2D → 4D, $\vec{\beta}$ still along $\hat{x} = \hat{x}'$:

$$\underbrace{\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix}}_{r'} = \underbrace{\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{call this } \Lambda} \underbrace{\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}}_{r} \qquad (1.12)$$

or $r' = \Lambda r$.

- If $\vec{\beta}$ is along $\hat{n}$ rather than $\hat{x}$:

$$r' = \Lambda_R^{-1} \Lambda \Lambda_R r \qquad (1.13)$$

where Λ_R is a <u>3D spatial rotation</u> which transforms the $\hat{n}$ direction to the $\hat{x}$ direction :

$$\Lambda_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ 0 & \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ 0 & \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{bmatrix} \qquad (1.14)$$

[for a rotation $(\Lambda_R^{-1})_{ji} = (\Lambda_R)_{ij}$, i.e. Λ_R is <u>orthogonal</u>].

- If $\vec{\beta}$ is along $-\hat{x}$ instead of x, change the sign of β in (1.12). That is,

if $\quad r' = \Lambda(\beta) r$ ← <u>direct</u> L.T.

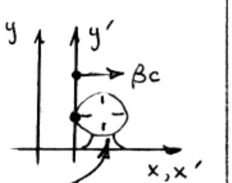
(Λ is a function of β)

then $\quad r = \Lambda(-\beta) r'$.

<u>inverse</u> Lorentz transformation.

1.7. TIME DILATION

As usual, S and S′ have the same spatial origin at $t = t' = 0$


clock at rest in S′

$$\Delta t' \equiv t_2' - t_1'$$

t_2': 2nd ring, t_1': 1st ring

as observed at <u>fixed x'</u>.

Using <u>inverse</u> Lorentz transformation,

$$ct_2 = \gamma c t_2' + \gamma\beta x_2' \qquad \text{but } x_2' = x_1'$$
$$ct_1 = \gamma c t_1' + \gamma\beta x_1' \qquad \text{(clock fixed in S')}$$

$$c\Delta t = \gamma c \Delta t', \qquad \boxed{\Delta t = \gamma \Delta t'} \qquad (1.15)$$

∴ Since γ always ≥ 1 the interval between rings is always <u>longer</u> when measured in a frame which is moving with respect to S′, where the two events occur at the <u>same place</u>.

Since S′ is a <u>unique</u> frame in which to observe the time interval between these two events, we assign a unique name to this time interval :

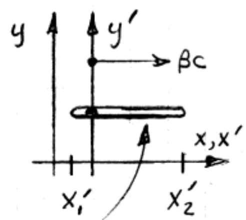
$$\Delta t' \equiv \Delta\tau \equiv \text{"proper time" interval}$$

$$\left.\begin{aligned} \Delta t &= \gamma \Delta\tau \\ dt &= \gamma d\tau \end{aligned}\right\} \text{"time dilation"} \qquad (1.16)$$

(Note that the observer in S uses his/her <u>own</u> fine grid of clocks and data loggers to measure Δt.)

1.8. SPACE CONTRACTION

$\Delta x' \equiv x_2' - x_1'$, measured at any t'.


rod at rest in S′

The observer in S, with his/her <u>own</u> fine grid of clocks, rulers, and data loggers, measures the positions x_1 and x_2 of the two ends of the rod at the <u>same time</u> $t_1 = t_2$.

Using <u>direct</u> L.T.,

$$x_2' = \gamma x_2 - \gamma\beta c t_2 \qquad \text{but } t_2 = t_1$$
$$x_1' = \gamma x_1 - \gamma\beta c t_1$$

$$\Delta x' = \gamma \Delta x \qquad \boxed{\Delta x = \Delta x'/\gamma} \qquad (1.17)$$

The length of the rod as observed in a system moving with respect to it is always <u>shorter</u> than its proper length $\Delta x'$.

The analysis of {1.7} could have been done with the direct L.T., and that of {1.8} with the inverse L.T. More algebra would be required to get the same result.

1.9. EINSTEIN LAW OF VELOCITY ADDITION

The identity

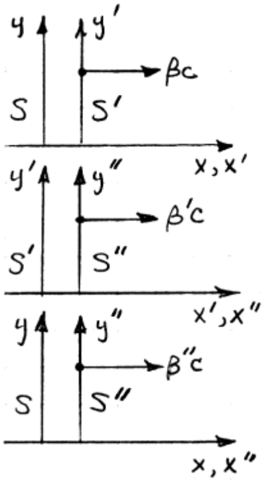
$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1+\tanh a \tanh b} \qquad (1.18)$$

can be believed in analogy to the well known

$$\tan(a+b) = \frac{\tan a + \tan b}{1 \ominus \tan a \tan b}$$

or it can be derived in a few lines using

$$\tanh a \equiv \frac{e^{a} - e^{-a}}{e^{a} + e^{-a}} . \qquad (1.19)$$



Problem: along the $\hat{x} = \hat{x}' = \hat{x}''$ direction,

S' has velocity	βc wrt	S
S''	$\beta' c$	S'
S''	$\beta'' c$	S

Given β and β', what is β''?

Since the boost parameter η is additive, we know

$$\eta'' = \eta + \eta'$$

But $\beta = \tanh \eta$ etc.

$$\therefore \quad \beta'' = \tanh \eta'' = \tanh(\eta + \eta')$$

$$= \frac{\tanh \eta + \tanh \eta'}{1 + \tanh \eta \tanh \eta'} \quad \text{using (1.18)}$$

$$\boxed{\beta'' = \frac{\beta + \beta'}{1 + \beta\beta'}} \qquad \text{(Einstein law)} \qquad (1.20)$$

1.10 SPACE TRAVEL: HUMAN CONSTRAINTS

Assume that an astronaut

- must be accelerated at $\leq 1g$
- must age ≤ 40 years during the voyage.

What maximum velocity can be achieved? How far can he/she travel, and how much time will have elapsed on earth?

The voyage consists of 10 years with $a_x' = +g$, 20 years $a_x' = -g$, 10 years $a_x' = +g$. We consider only the first leg. To get the answers to the questions above, we then double the first leg distance and quadruple the first leg time.

Suppose that the astronaut at a certain moment has an $\hat{x}$ velocity equal to βc. The astronaut is accelerating and so his/her rest frame is not inertial. To analyze his/her motion using the Lorentz transformation we need an inertial frame, so we define a comoving frame S' which is instantaneously at rest with respect to the astronaut but which is not accelerating.

In an infinitesimal proper time interval $d\tau$ (= same in astronaut and comoving frames, since relative $\gamma_{rel} = 1$ to 2nd order in β_{rel}), the astronaut's velocity increases, relative to comoving frame, by $g\,d\tau$:

$$g\,d\tau = dv_{rel} \equiv c\,d\beta_{rel}$$

Since $\beta_{rel} = 0$ and $d\beta_{rel}$ is infinitesimally small,

$$d\eta_{rel} \cong d\beta_{rel}$$

where η is the boost parameter. Since the boost parameter is additive, as seen on the earth (frame S)

$$\eta(\tau + d\tau) = \eta(\tau) + d\eta_{rel}$$

$$d\eta / d\tau \cong d\beta_{rel}/d\tau = g/c$$

$$\eta_{max} = \int_0^{\tau_0 = 10\,yrs} \frac{d\eta}{d\tau}\, d\tau$$

$$= \int_0^{\tau_0} \frac{g}{c}\, d\tau = \frac{g\tau_0}{c} = 10.34^{*}$$

$$\beta_{max} = \tanh \eta_{max} = 1 - (2.09 \times 10^{-9}) .$$

* The most boosted particles in accelerators (electrons at LEP) have $\eta \approx 12.2$.

The distance covered is

$$dx = \beta c\, dt = (\tanh\eta) c(\gamma d\tau) \quad \text{using time dilation}$$
$$= c(\tanh\eta)(\cosh\eta)\, d\tau$$
$$= c \sinh\eta\, d\tau$$
$$\Delta x = \underbrace{2}_{\text{2 legs}} c\int_0^{\tau_0} \sinh\eta\, d\tau = 2c\int_0^{\tau_0} \sinh\left(\frac{g\tau}{c}\right) d\tau$$
$$= 2\frac{c^2}{g}\left(\cosh\frac{g\tau_0}{c} - 1\right) \quad \text{meters}$$
$$= 2.84 \times 10^{20} \text{ meters}$$

$= 29,900$ light years, or $\approx 2\times10^{-7}$ the size of the universe. So only $\approx 10^{-20}$ of it can be explored by man.

The time elapsed on earth is

$$dt = \gamma d\tau = \cosh\eta\, d\tau$$
$$\Delta t = \underbrace{4}_{\text{4 legs}}\int_0^{\tau_0} \cosh\frac{g\tau}{c}\, d\tau = 4\frac{c}{g}\sinh\frac{g\tau_0}{c}$$
$$= 1.89 \times 10^{12} \text{ sec}$$
$$= 59,850 \text{ yrs} \quad \text{(compare 40 yrs.)}$$

This last result is called the "twin paradox." It is not a paradox because the earthbound twin is not accelerating.

1.11 FOUR-MOMENTUM

If we wish to write Eq. (1.2) in the form

$$(r_B - r_A)^2 = c^2(t_B - t_A)^2 - (x_B - x_A)^2 - (y_B - y_A)^2 - (z_B - z_A)^2$$
$$\equiv r_B{}^2 - 2 r_B \cdot r_A + r_A{}^2$$

with $r_B \cdot r_A \equiv$ inner product of 2 4-vectors in spacetime, it must be the case that

$$r_B \cdot r_A \equiv c^2 t_B t_A - x_B x_A - y_B y_A - z_B z_A$$

and that the inner product of any 2 4-vectors is independent of reference frame (invariant to Lorentz transformations).

The proper time interval $d\tau$ and the rest mass m are also Lorentz invariants.

Form $p \equiv m\dfrac{dr}{d\tau}$, that is

$$(p_0, p_x, p_y, p_z) = \left(mc\frac{dt}{d\tau}, m\frac{dx}{d\tau}, m\frac{dy}{d\tau}, m\frac{dz}{d\tau}\right).$$

Note that $dt/d\tau = \gamma$ so $dx/d\tau = \gamma\, dx/dt$. So

$$p = (\gamma mc, \gamma m v_x, \gamma m v_y, \gamma m v_z) \tag{1.21}$$
$$\equiv (E/c, \vec{p})$$

must transform like r, i.e. must also be a 4-vector. It is called the four-momentum. We can write

$$\begin{bmatrix} E'/c \\ p_x' \\ p_y' \\ p_z' \end{bmatrix} = \Lambda \begin{bmatrix} E/c \\ p_x \\ p_y \\ p_z \end{bmatrix} \quad \text{with } \Lambda \text{ as in (1.12).}$$

The length^2 of p is Lorentz invariant and we can evaluate it in a frame in which the CM of the system it describes is not moving ($\vec{p} = 0$, $\gamma = 1$). Then

$$p^2 = \underbrace{E^2/c^2 - |\vec{p}|^2}_{\text{true in any frame}} = \underbrace{m^2c^2}_{\text{rest frame value}} \tag{1.22}$$

This is the fundamental equation for solving relativistic kinematics problems.

What is E? Make a Taylor series expansion

$$E = \frac{mc^2}{(1 - v^2/c^2)^{1/2}} = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right)$$

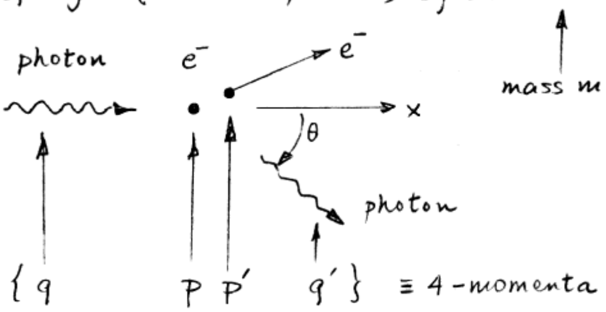
For $v \ll c$ this is $E = mc^2 + \frac{1}{2}mv^2$ where the last term is the nonrelativistic kinetic energy. We interpret the first term as the rest mass energy:

$$\underbrace{E}_{\text{total energy}} = \gamma mc^2 = \underbrace{mc^2}_{\text{rest mass energy}} + \underbrace{T}_{\text{kinetic energy}} \tag{1.23}$$

We see the possibility of converting mass to energy (lots of energy because c^2 is large).

1.12 COMPTON (PHOTON–ELECTRON) SCATTERING

To illustrate the power of Eq. (1.22) for solving problems in relativistic kinematics, we consider the scattering of a quantum of light (massless photon) by an electron at rest.



$p = (mc, \vec{0})$ because target electron at rest.

$q = (q_0, q_x, 0, 0)$

Since photon massless, $q \cdot q = 0$ by Eq. (1.22), so $q_x = q_0$ and we can write

$$q = (q_0, q_0, 0, 0); \quad q' = (q_0', q_0'\cos\theta, q_0'\sin\theta, 0)$$

Problem: what is the relationship between the final photon energy cq_0' and its final angle θ wrt $\hat{x}$?

Use energy-momentum conservation $\Rightarrow$ 4-momentum conservation:

$$q + p = q' + p' \quad \text{(this is 4 equations!)}$$

$$q - q' + p = p'$$

$$[(q-q') + p] \cdot [(q-q') + p] = p' \cdot p'$$

$$(q-q')\cdot(q-q') + 2p\cdot(q-q') + p\cdot p = p'\cdot p'$$

$$\underset{0}{\underset{\parallel}{q\cdot q}} - 2q\cdot q' + \underset{0}{\underset{\parallel}{q'\cdot q'}} + 2p\cdot(q-q') + \underset{\cancel{(mc)^2}}{\underset{\parallel}{p\cdot p}} = \underset{\cancel{(mc)^2}}{\underset{\parallel}{p'\cdot p'}}$$

$$\cancel{2}p\cdot(q-q') = \cancel{2}q\cdot q'$$

$$(mc, 0, 0, 0)\cdot(q_0 - q_0', q_0 - q_0'\cos\theta, -q_0'\sin\theta, 0) =$$

$$= (q_0, q_0, 0, 0)\cdot(q_0', q_0'\cos\theta, q_0'\sin\theta, 0)$$

$$mc(q_0 - q_0') = q_0 q_0'(1-\cos\theta) \qquad \div\, q_0 q_0' mc:$$

$$\boxed{\frac{1}{q_0'} - \frac{1}{q_0} = \frac{1}{mc}(1-\cos\theta)} \qquad (1.24)$$

This is A.H. Compton's famous formula. Conventionally it is multiplied by Planck's constant h, with the photon wavelength $\lambda = h/q_0$. Then

$$\boxed{\lambda' - \lambda = \lambda_c(1-\cos\theta)} \text{ with} \qquad (1.25)$$

$$\lambda_c \equiv h/mc\,.$$

λ_c, the Compton wavelength of the electron, is

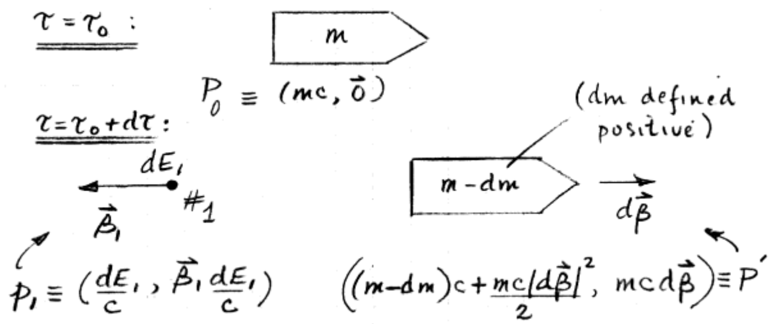
$$\lambda_c = 2\pi \times 386 \times 10^{-15}\, m$$

Planck's constant is

$$h = 2\pi \times 6.58 \times 10^{-16} \text{ eV sec}.$$

1.13 SPACE TRAVEL: PROPULSION CONSTRAINTS

Again consider the spacecraft as viewed in the comoving frame (§1.10). In infinitesimal proper time interval $d\tau$ the rocket motor ejects particle #1 with energy dE_1 and relative velocity $\beta_1 c$.



In assigning the 4-momentum to particle #1, we used $p = (\gamma mc, \gamma\vec{\beta} mc)$ so that $\vec{p}/p_0 = \vec{\beta}$. In assigning the 4-momentum to the spacecraft, we used the fact that (as viewed in the comoving frame) the spacecraft is nonrelativistic, so that $E \approx mc^2 + \frac{1}{2}mv^2$.

If we assume a perfectly efficient engine, i.e. no heat energy radiated in random directions, both energy and momentum will be conserved:

$$P_0 = p_1 + P'$$

neglect, 2nd order in infinitesimals.

$$\cancel{mc} = \frac{dE_1}{c} + (\cancel{m} - dm)c + \cancel{\frac{mc}{2}|d\vec{\beta}|^2} \qquad \text{(timelike part)}$$

$$\vec{0} = \vec{\beta}_1 \frac{dE_1}{c} + mc\,d\vec{\beta} \qquad \text{(spacelike part)}$$

Substituting $\frac{dE_1}{c} = c\,dm$ from the timelike eq$^{\underline{n}}$, the spacelike eq$^{\underline{n}}$ becomes

$$|d\vec{\beta}| = |\vec{\beta}_1| \frac{dm}{m}.$$

Again we set $|d\vec{\beta}| \approx d\eta$, where η is the boost, since rocket is nonrelativistic in comoving frame.

As additional particles (#2, #3, etc) are ejected, the boosts $d\eta_i$ are <u>additive</u>.

$$\therefore \quad \eta_{final} - (\eta_0 \equiv 0) = \int_{m_0}^{m_{final}} \beta_1 \frac{|dm|}{m}$$

$$\boxed{\eta_{final} = \beta_1 \ln \frac{m_0}{m_{final}}} \qquad (1.26)$$

<u>Chemical rocket engines</u> achieve maximum $\beta_1 \approx 4 \times 10^3$ m/sec$/c \approx 1.33 \times 10^{-5}$. Then to achieve a boost of 10.34 (see §1.10) requires

$$\ln \frac{m_0}{m_f} = 7.8 \times 10^5$$

$$m_f = m_0 \times (\text{a number beyond calculator range}).$$

Chemical engines <u>will not suffice</u>.

<u>Relativistic engines</u> emit particles at $\beta_1 \cong 1$. If they were unit efficient,

$$\ln \frac{m_0}{m_f} = 10.34$$

$$m_f = 3.1 \times 10^4 \, m_0 .$$

<u>Manned payload</u> requires $m_f \geq 10T$ for life support; then

$$m_0 \geq 3.1 \times 10^5 \, T$$

⇒ a rocket heavier than an aircraft carrier. ($\leq 10^5 T$)

Note that Eq. (1.26) becomes

$$\eta_{final} = \underline{\epsilon} \beta_1 \ln \frac{m_0}{m_{final}} \qquad (1.27)$$

if the efficiency ϵ of the engine is not unity.

Present relativistic engine concepts...

- are grossly inefficient ($\epsilon \ll 1$)
- leave most of their fuel on board so that m_f/m_0 cannot be $\ll 1$.

(Example: laser powered by batteries)

1.14 OTHER FOUR-VECTORS

In addition to

$$r = (ct, \vec{r})$$

$$p = (E/c, \vec{p}) \qquad (E \equiv \gamma mc^2, \ \vec{p} \equiv \gamma m\vec{v}),$$

frequently encountered other 4-vectors are

$$\partial \equiv \left(\frac{\partial}{c\partial t}, -\vec{\nabla}\right) \qquad \left(\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$$

$$k \equiv \left(\frac{\omega}{c}, \vec{k}\right) \quad \text{as in } e^{i(\omega t - \vec{k}\cdot\vec{r})} \qquad (1.28)$$

"wave vector"

ω = angular freq.

$$A \equiv (\varphi, \vec{A}) \qquad (1.29)$$

"vector potential"

$$\text{where} \begin{cases} \vec{B} \equiv \vec{\nabla} \times \vec{A} \\ \vec{E} \equiv -\vec{\nabla}\varphi - \dfrac{\partial \vec{A}}{c\partial t} . \end{cases}$$

Because dot products of 4-vectors are Lorentz invariant, so are

$$k \cdot r \equiv \omega t - \vec{k}\cdot\vec{r} \quad \text{"phase of a wave"}$$

$$\partial \cdot \partial \equiv \Box = \frac{\partial^2}{c^2 \partial t^2} - \nabla^2 \quad \text{"D'Alembertian"}$$

$$\partial \cdot A \equiv \frac{\partial \phi}{c\partial t} + \vec{\nabla}\cdot\vec{A}$$

(= 0 when $\vec{A}$ satisfies the "Lorentz gauge condition")

When the "de Broglie momentum" equation $|\vec{p}| = h/\lambda$ is combined with the "Planck frequency" equation $E = h\nu$, using

$$|\vec{k}| \equiv 2\pi/\lambda , \qquad (1.30)$$

both equations can be summarized by

$$(E/c, \vec{p}) \equiv \boxed{p = \frac{h}{2\pi} k} \equiv \frac{h}{2\pi}\left(\frac{\omega}{c}, \vec{k}\right) \qquad (1.31).$$

(this is 4 equations)

"generalized de Broglie eqn"

Another 4-vector is

$$j \equiv (c\rho, \vec{J}) \qquad \begin{matrix} \rho = \text{chg density (esu/cm}^3) \\ \vec{J} = \text{current density (esu/cm}^2\text{-sec)} \end{matrix}$$

Lorentz invariant $\Big\}$ $$\partial \cdot j = \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \qquad (1.32)$$

(charge conservation)

1.15 LORENTZ TRANSFORMATION OF ELECTROMAGNETIC FIELDS

The fact that

$$\begin{bmatrix}\phi'\\ A_x'\\ A_y'\\ A_z'\end{bmatrix} = \begin{bmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\phi\\ A_x\\ A_y\\ A_z\end{bmatrix}$$

and

$$\left.\begin{aligned}\vec{B} &= \vec{\nabla}\times\vec{A}\\ \vec{E} &= -\vec{\nabla}\phi - \frac{\partial\vec{A}}{c\,\partial t}\end{aligned}\right\}\ \text{cgs units!}$$

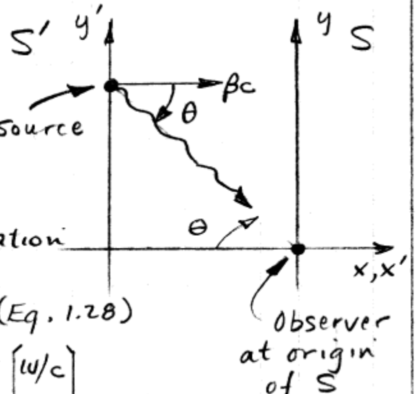
leads after some algebra to the following transformation equations for $\vec{E}$ and $\vec{B}$:

$$\begin{aligned}\vec{E}_\perp' &= \gamma(\vec{E}_\perp + \vec{\beta}\times\vec{B})\\ \vec{B}_\perp' &= \gamma(\vec{B}_\perp - \vec{\beta}\times\vec{E})\\ \vec{E}_{/\!/}' &= \vec{E}_{/\!/},\ \vec{B}_{/\!/}' = \vec{B}_{/\!/}\end{aligned} \qquad (1.33)$$

where "$\perp$" means $\perp$ to $\hat{\beta}$ and "$/\!/$" means parallel to $\hat{\beta}$.

A consequence of Eq. (1.33) is that $|\vec{E}|^2 - |\vec{B}|^2$ is a Lorentz invariant.

1.16 RELATIVISTIC DOPPLER SHIFT



Apply the direct Lorentz transformation (Eq. 1.12) to the wave 4-vector k (Eq. 1.28)

$$\begin{bmatrix}\omega'/c\\ k_x'\\ k_y'\\ k_z'\end{bmatrix} = \Lambda\begin{bmatrix}\omega/c\\ k_x\\ k_y\\ k_z\end{bmatrix}$$

$$\Rightarrow\ \frac{\omega'}{c} = \gamma\frac{\omega}{c} - \gamma\beta k_x \qquad (1.34)$$

Let the lab phase velocity of the wave be $\beta_s c$ ($\beta_s = 1$ for a light wave). ($T \equiv$ period)

$$|\vec{k}| = \frac{2\pi}{\lambda} = \frac{2\pi}{\beta_s c T} = \frac{2\pi\nu}{\beta_s c} = \frac{\omega}{\beta_s c}$$

so in Eq. (1.34) we may write

$$k_x = |\vec{k}|\cos\theta = \frac{\omega}{\beta_s c}\cos\theta. \quad \text{Then}$$

$$\omega' = \gamma\omega\left(1 - \frac{\beta}{\beta_s}\cos\theta\right)$$

$$\boxed{\omega = \frac{\omega'}{\gamma\left(1 - \frac{\beta}{\beta_s}\cos\theta\right)}} \quad \text{Relativistic Doppler shift} \qquad (1.35)$$

Special cases:

- light wave $\Rightarrow \beta_s = 1$

$$\boxed{\omega = \frac{\omega'}{\gamma(1 - \beta\cos\theta)}} \qquad (1.36)$$

- $\left\{\begin{array}{ll}\text{approaching} & (\theta = 0)\\ \text{receding} & (\theta = \pi)\end{array}\right\}$ light wave:

$$\omega = \frac{\omega'}{\gamma(1 \mp \beta)} = \left(\frac{1 \pm \beta}{1 \mp \beta}\right)^{1/2}\omega'$$

- $\cos\theta = 0$ (source is at zenith, where nonrelativistically there is no Doppler shift):

$$\omega = \frac{\omega'}{\gamma},\quad T = T'\gamma \quad \left(\begin{array}{l}\text{ordinary}\\ \text{time}\\ \text{dilation}\end{array}\right)$$

- $\beta \ll 1$

$$\omega \cong \frac{\omega'}{1 - \frac{\beta}{\beta_s}\cos\theta} = \frac{\omega'}{1 - \frac{v_{source}}{v_{wave}}\cos\theta}$$

(= freshman physics Doppler shift. Note sonic boom at $\cos\theta = v_{wave}/v_{source}$.)