Problem #1
10 Points

Problem Set #4
NE 290H, Barnard and Lund
Due Feb. 16, 2009

S.M. Lund P6/

6/ TPE Problem 6

Show that the principal functions of the transfer matrix solution of the particle orbit

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s|s_i) \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix} = \begin{pmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{pmatrix} \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}$$

are:

$$C(s|s_i) = \frac{w(s)}{w_i} \cos \Delta\psi(s) - w_i' w(s) \sin \Delta\psi(s)$$

$$S(s|s_i) = w_i w(s) \sin \Delta\psi(s)$$

$$C'(s|s_i) = \left(\frac{w'(s)}{w_i} - \frac{w_i'}{w(s)} \right) \cos \Delta\psi(s) - \left(\frac{1}{w_i w(s)} + w_i' w'(s) \right) \sin \Delta\psi(s)$$

$$S'(s|s_i) = \frac{w_i}{w(s)} \cos \Delta\psi(s) + w_i w'(s) \sin \Delta\psi(s)$$

$$\Delta\psi(s) = \psi - \psi_i = \int_{s_i}^{s} \frac{d\bar{s}}{w^2(\bar{s})}$$

$$w_i = w(s = s_i)$$
$$w_i' = w'(s = s_i)$$

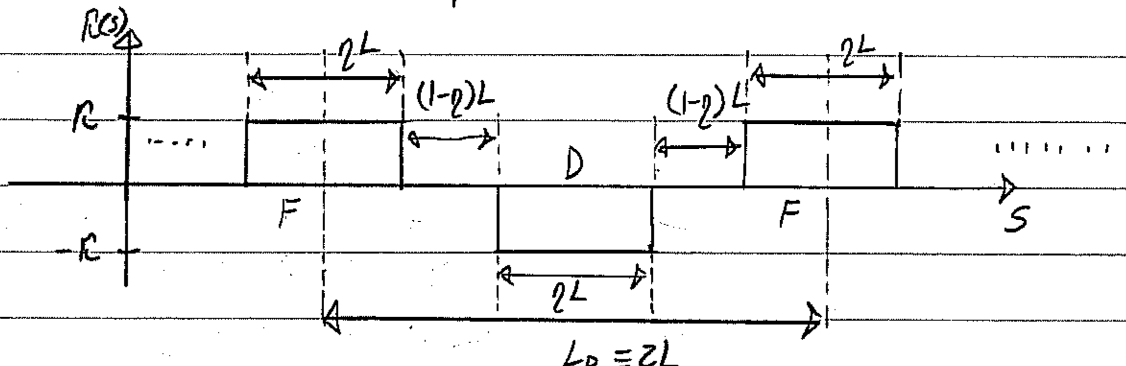
Hint use:

$$x = A_i w \cos\psi$$
$$x' = A_i w' \cos\psi - \frac{A_i}{w} \sin\psi$$

$$\psi = \psi_i + \Delta\psi$$

TPE Problem 7 | Problem #2 20 Points | S.M. Lund P7/1

7/1 Consider a "FODO" periodic lattice:



$L_p = 2L$ = lattice period

ηL = Quadrupole lengths

$(1-\eta)L$ = drift lengths

η = Quadrupole occupancy $0 < \eta \leq 1$

κ = Quadrupole strength

a) Write the transfer matrices $\bar{M}(s|s_i)$ for each section of the periodic lattice.
In terms of $\theta \equiv \sqrt{\kappa}\,\eta L$, d, and η. Use results from problem set #1.

$\bar{M}_F$: Transfer through Focus Quadrupole.

$\bar{M}_O$: " " Drift

$\bar{M}_D$: " " Defocus Quadrupole

$\bar{M}_O$: " " Drift.

b) Write the transfer matrix $\bar{M}(s_i+L_p, s_i)$ through one lattice period starting from the left side of a focus quadrupole. No need to fully expand!

c) Show that the phase advance σ_0 of a particle through this lattice period

$$\cos\sigma_0 = \frac{1}{2}\,\mathrm{Trace}\, M(s_i + L_p | s_i)$$

can be expressed as:

$$\cos\sigma_0 = \cos\Theta\cosh\Theta + \frac{(1-\eta)}{\eta}\Theta(\cos\Theta\sinh\Theta - \sin\Theta\cosh\Theta)$$
$$-\frac{1}{2}\frac{(1-\eta)^2}{\eta^2}\Theta^2\sin\Theta\sinh\Theta$$

d) Will it matter where the lattice period is started in the calculation of σ_0 in part c)? Why?

e) For $\Theta \ll 1$ (thin lens limit), show that

$$\cos\sigma_0 \simeq 1 - \frac{1}{2}\left(1 - \frac{2}{3}\eta\right)\frac{\Theta^4}{\eta^2}$$

f) If $\sigma_0 \ll 1$, and $\eta \ll 1$, show that

$$\sigma_0 \simeq \eta|\hat{\kappa}|L^2$$

g) If one wanted to model a "FODO" focusing lattice by a continuous focusing channel with $\kappa(s) = k_{\beta0}^2 = \text{const.}$, how could one choose $k_{\beta0}^2$ based on part f)?

TPE Problem 9

Problem #3
15 Points

S. M. Lund

9/ In class we derived the single-particle Courant-Snyder Invariant:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon = \text{const.}$$

where:

$$\beta(s) = w^2(s)$$

$$\alpha(s) = -w(s) w'(s)$$

$$\gamma(s) = \frac{1}{w^2(s)} + w'(s)^2 = \frac{1+\alpha^2(s)}{\beta(s)}$$

Derive the critical values of the ellipse indicated on the figure below:

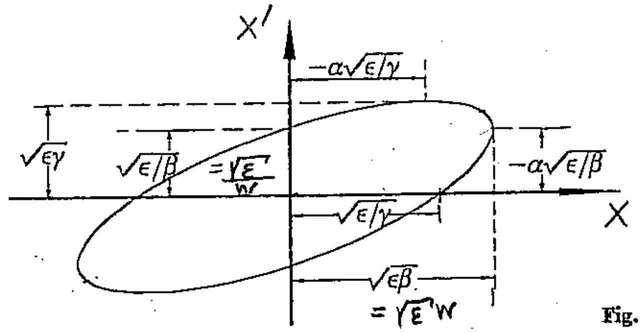


Fig. 5.22. Phase space ellipse

From Wiedemann

Hint: to avoid messy algebra, take a differential of the constraint equation $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \text{const}$ and use this result to find turning points.

$$\Rightarrow \quad 2\gamma x\, dx + 2\alpha x\, dx' + 2\alpha x'\, dx + 2\beta x'\, dx' = 0$$

These results are important in understanding the KV distribution derived later to model beams with space-charge

TPE Problem 10 | Problem # 4 10 Points | S. M. Lund P10/

10/ Bends

Part I - Magnetic Bends

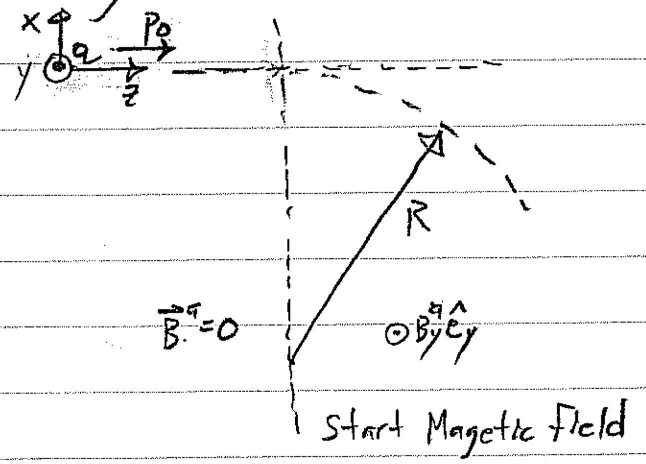
a) From the Lorentz Force equation show that a static magnetic field $\vec{B}_q$ cannot change the kinetic energy of a particle $\mathcal{E} \equiv (\gamma - 1) m c^2$

$$m \frac{d}{dt}(\gamma \vec{\beta}) = q \vec{\beta} \times \vec{B}_q$$

$$\gamma = \frac{1}{\sqrt{1-\vec{\beta}^2}} \qquad \vec{\beta} = \frac{1}{c}\frac{d\vec{x}}{dt}$$

$$\circ \equiv \frac{d}{dt}$$

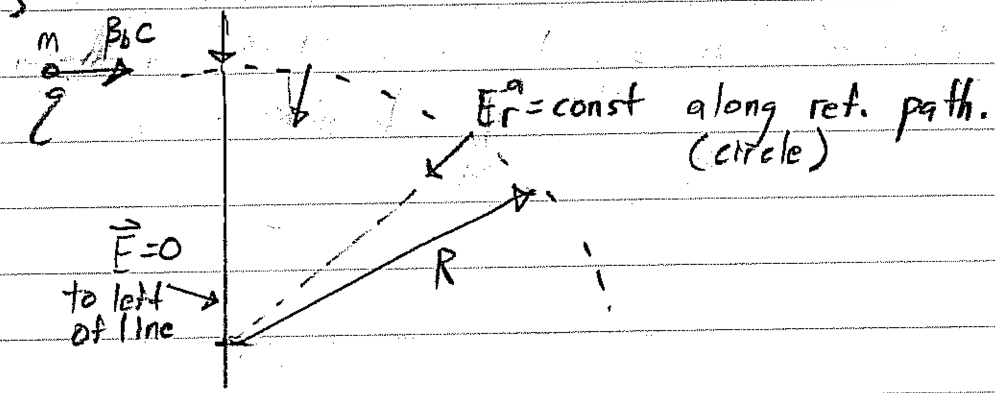
b) Using the result of part a) derive the formula connecting the bend radius R for a particle with momentum $p_0 = mc\gamma_b \beta_b = \text{const}$ entering a uniform magnetic field $\vec{B}_q = B_y^q \hat{y} = \text{const}$. You can assume that the orbit is circular in the magnetic field and use the result in a).



TPE Problem 10

Part II

Rather than with magnetic fields, bends can be implemented using radial electric fields. Ideally,



Derive a formula relating E_r^a and p_0 to the bend radius R. Explain how this might be implemented in with an electric optic - i.e., what configuration of plates and voltages can be used to realize the electric dipole bend.?

TPE Problem 11 | Problem #5 15 Points | S. M. Lund | P11/

11/ Dispersion Function:

Consider the single-particle dispersion function:

$$D'' + \kappa_x D = \frac{1}{R(s)}$$

Part I

Calculate the evolution of D' and D from an initial condition

$$D(s=s_i) = D_i$$
$$D'(s=s_i) = D_i'$$

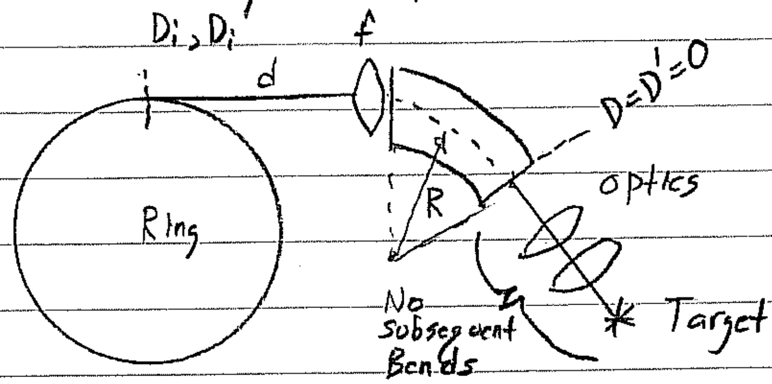
a) in a drift section ($\kappa_x = 0$, $R \to \infty$)

b) as a result of a thin-lens focusing kick at $s = s_i$ with focal length f ($\kappa_x = \frac{1}{f}\delta(s=s_i)$ and $R \to \infty$) and the initial coordinates are at $s = s_i^-$ and the final coordinates are at $s = s_i^+$

c) in a bend ($\kappa_x = 0$, $R = \text{const} \neq 0$)

Part II

A particle is kicked out of a ring with dispersion $D' = D_i'$ and $D = D_i$ just after the kick and then through an extraction line with a drift of length d, a thin lens focus kick with focal length f, then a bend of length ℓ and radius R, and finally a series of final optics to the target.



TPE Problem 11

S. M. Lund p11/

What constraints among the lattice parameters d, f, R, and ℓ can be enforced to ensure that $D = 0 = D'$ after the magnet in the transport line leading to the target? Are these constraints practical to implement (qualitative answer only)?