Problem # 1
30 points

~~PRS Problem 1~~

Problem Set #12
NE290H, Barnard and Lund
Due May 4, 2009

1/ Moment Equations and Conservation Constraints

The nonrelativistic Vlasov equation is:

$$\left\{ \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} + \frac{q}{m}\left[\vec{E} + \vec{v} \times \vec{B}\right] \cdot \frac{\partial}{\partial \vec{v}} \right\} f(\vec{x}, \vec{v}, t) = 0$$

Define a fluid density n and a fluid flow velocity $\vec{V}$ by

$$n(\vec{x}, t) = \int d^3v \cdot f(\vec{x}, \vec{v}, t)$$

$$n(\vec{x}, t)\vec{V}(\vec{x}, t) = \int d^3v \; \vec{v} f(\vec{x}, \vec{v}, t)$$

a) Operate on the Vlasov equation with

$$\int d^3v \ldots\ldots$$

to derive the continuity equation:

$$\frac{\partial}{\partial t} n(\vec{x}, t) + \frac{\partial}{\partial \vec{x}} \cdot \left(n(\vec{x}, t) \vec{V}(\vec{x}, t) \right) = 0$$

b) Can the continuity equation be solved by itself if you specify the initial density field $n(\vec{x}, t=0)$? Why?

c) Operate on Vlasov's equation with

$$\int d^3v \; \vec{v} \ldots\ldots$$

to derive the fluid force equation.

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$$\frac{\partial}{\partial t}(n\vec{V}) + \nabla\cdot(n\langle\vec{v}\vec{v}\rangle_v) = \frac{q}{m}n(\vec{E}+\vec{V}\times\vec{B})$$

$$\langle\vec{v}\vec{v}\rangle_v \equiv \int d^3v\, \vec{v}\vec{v} f \Big/ \int d^3v\, f$$

John Barnard in earlier lectures made a definition of a pressure tensor as

$$\underline{\underline{P}} = m\int d^3v\, (\vec{v}-\vec{V})(\vec{v}-\vec{V}) f(\vec{x},\vec{v},t)$$
$$= mn\langle\vec{v}\vec{v}\rangle_v - mn\vec{V}\vec{V}$$

in terms of this the fluid force eqn can be expressed as:

$$\frac{\partial}{\partial t}\vec{V} + \vec{V}\cdot\frac{\partial}{\partial\vec{x}}\vec{V} = \frac{q}{m}(\vec{E}+\vec{V}\times\vec{B}) - \frac{1}{mn}\frac{\partial}{\partial\vec{x}}\cdot\underline{\underline{P}}$$

This form is often used in fluid/plasma analysis.

d) If the continuity and force equation derived in parts a) and c) are analyzed, can they be solved in principle if you specify the initial density field $n(\vec{x},t=0)$ and the velocity field $\vec{V}(\vec{x},t=0)$? Why? Does the answer change if we assume a cold initial beam with $\underline{\underline{P}} = 0$? Why?

e) Let $G(f)$ be some smooth, differentiable function of f satisfying $G(f\to 0) = 0$. Show that

$$\int d^3x \int d^3v\, G(f) = \text{const.}$$

This so-called "generalized entropy" measure (with G specified) can be used to check Vlasov simulations. For example:

$$G(f) = f: \quad \int d^3x \int d^3v\, f = \text{const} \Rightarrow \text{charge cons.}$$

$$G(f) = f^2: \quad \int d^3x \int d^3v\, f^2 = \text{const} \Rightarrow \text{"enstropy" cons.}$$

$\vdots$

TKS Problem 4

Problem #2
10 points

Problem KV Characteristics

An orbit in a continuously focused KV beam satisfies:

$$\tilde{\vec{x}}_\perp'' + k_{\beta 0}^2 \tilde{\vec{x}}_\perp - \frac{Q}{r_b^2}\tilde{\vec{x}}_\perp = 0$$

$$\tilde{\vec{x}}_\perp'' + k_\beta^2 \tilde{\vec{x}}_\perp = 0$$

$$k_\beta^2 = k_{\beta 0}^2 - \frac{Q}{r_b^2}$$

r_b = beam radius

Q = perveance

$k_{\beta 0}^2$ = focusing wavenumber squared.

$\tilde{x}, \tilde{y}$ = x, y coords of characteristic orbits.

Show that the orbit characteristics can be expressed as

$$\tilde{r}^2(\tilde{s}) = r^2\cos^2[k_\beta(\tilde{s}-s)] + \frac{rr'}{k_\beta}\cos\psi\,\sin[2k_\beta(\tilde{s}-s)] + \frac{r'^2}{k_\beta}\sin^2[k_\beta(\tilde{s}-s)]$$

$$\psi = \theta - \theta'$$

$$\tilde{x}(\tilde{s}=s) = r\cos\theta \qquad \tilde{x}'(\tilde{s}=s) = r'\cos\theta'$$

$$\tilde{y}(\tilde{s}=s) = r\sin\theta \qquad \tilde{y}'(\tilde{s}=s) = r'\sin\theta'$$

Angles in polar coordinates.

Problem #3
20 points

TKS Problem 2

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Gluckstern Modes on a KV Beam

2/ $n=1$ Gluckstern mode and the KV envelope equation for the breathing mode.
r_b = equilibrium matched beam radius.

a) The Gluckstern mode eigenfunction is given by

$$\delta\phi_n = \begin{cases} \frac{A_n}{2}\left[P_{n-1}\left(1-\frac{2r^2}{r_b^2}\right) + P_n\left(1-\frac{2r^2}{r_b^2}\right)\right] ; & 0 \le r < r_b \\ 0 & r_b < r \le r_p \end{cases}$$

$n = 1, 2, 3, \ldots$; $P_n(x)$ = nth order Legendre Polynomial

Write down the eigenfunction as an explicit polynomial in r for $n=1$ and plot this solution.

Legendre Polynomials

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$

$\vdots \quad \vdots$

b) Apply the Poisson equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\delta\phi_n}{\partial r}\right) = \frac{-q}{\epsilon_0}\delta n_n(r).$$

to calculate the perturbed mode density δn_n for $\delta\phi_1$ as a function of r for $0 \le r < r_b$. (the "body-wave" component). Plot this result.

c) Use part b) to calculate the amount of charge introduced into the system by the "body-wave" perturbation $\delta n_1(r)$ for $0 \le r < r_b$. How far would the beam edge radius $r_e = r_b + \delta r_b$ need to change to conserve charge to linear order in A_1?

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d) Obtain the $n=1$ Gluckstern mode dispersion relation from the general n formula presented in class:

$$2n + \frac{1-(\sigma/\sigma_0)^2}{(\sigma/\sigma_0)^2}\left[B_{n-1}\left(\frac{k/k_{\beta 0}}{\sigma/\sigma_0}\right) - B_n\left(\frac{k/k_{\beta 0}}{\sigma/\sigma_0}\right)\right] = 0$$

From the definitions in the class notes for the B_n we have:

$$B_0(\alpha) = 1 \qquad B_1(\alpha) = \frac{(\alpha/2)^2}{(\alpha/2)^2 - 1}$$

Solve for the mode eigen"frequency" k as a function of $k_{\beta 0}$ and σ/σ_0.

k is a spatial wavenumber that we sometimes call a "frequency"

e) Compare the wavenumber k calculated in part d) with the "breathing" envelope mode on a round KV equilibrium where we showed that the mode wavenumber is

$$k_{evelope} = \sqrt{2k_{\beta 0}^2 + 2k_{\beta 0}^2(\sigma/\sigma_0)^2}$$

Are the wavenumbers the same? Is it reasonable to identify these as the same modes? (Explain why.) Would you expect that the lowest order modes of a kinetic theory to always reproduce the KV envelope modes to lowest order? (Explain why)