Magneto-mechanics of wires and shells for algebraic motion of vertically displacing plasmas

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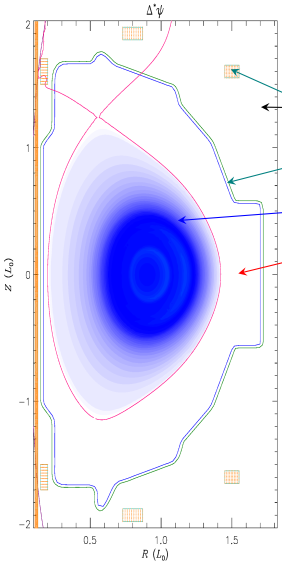
Complexity of the physics involved in VDEs



1. Multi-domain, geometrical/topological 3D problem
   - vacuum + boundary → harmonic potentials
   - first wall, vessel and external coils → Darwin approximation of Maxwell
   - plasma → extended-MHD
   - halo region → (?)
2. highly non-linear, advection-diffusion problem
   - regimes featuring hyperbolic / parabolic / elliptic operators
   - mixing of time-scales
   - stiffness (strong influence of parameters)
3. time-dependent and externally driven problem
   - loop voltage, feedback control, external coils, sources/sinks
   - never in equilibrium (linear theory not helpful)

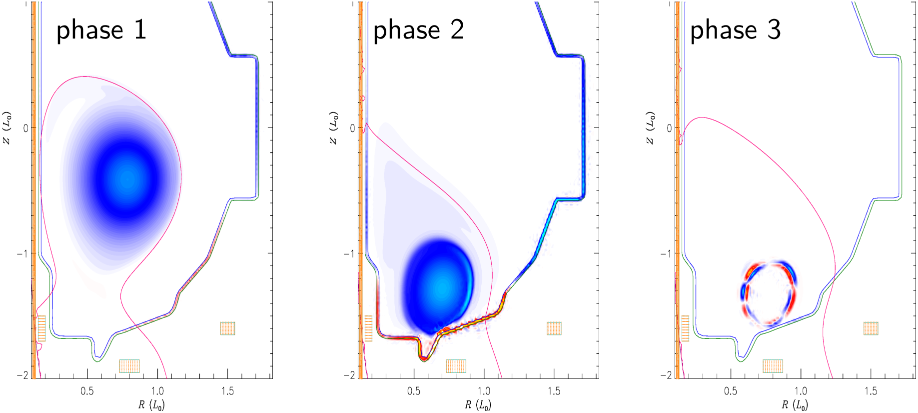
Modelling efforts

Driving questions: how well can we

1. model ?
2. understand ?
3. predict ?

- flexible geometry (3D) and topology (holes, gaps, cuts)
  $\Rightarrow$ solution space, boundary conditions
- stability in space and time :
  - discretisation/mesh versus fine-scale structures (current sheets)
    $\Rightarrow$ limits in Lundquist numbers, gradient scale-lengths
  - numerical modes versus ideal/resistive instabilities
    $\Rightarrow$ limits in time-step compared to Alfvén frequencies
  - valid throughout various PDE/ODE regimes
    $\Rightarrow$ limits in parameter space, stiffness, sensitivity, numerical scheme

Modelling of NSTX#132859 with M3DC1 [Ferraro and Jardin, 2009]



vertical (resistive) instability

- diverted plasma (no contact with wall)
- downwards drift (advection)
- inductive coupling

first contact with wall

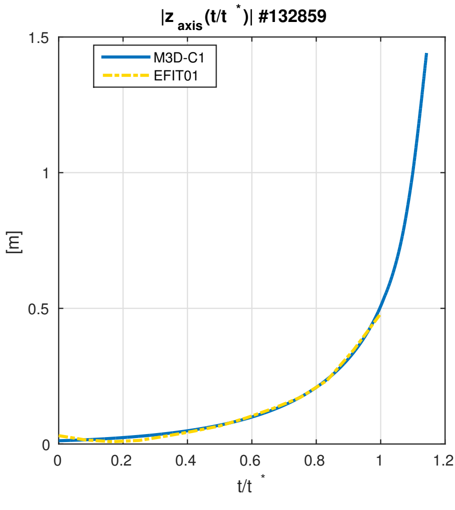
- limited plasma by wall
- scrape-off of LCFS
- shared currents
- mainly 2D, $q_{\text{edge}} > 2$

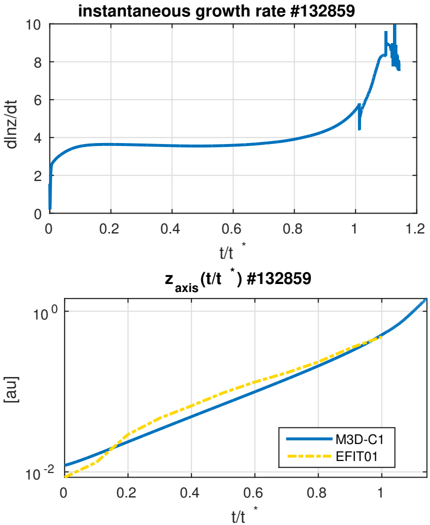
MHD (kink) unstable

- 3D wall/halo currents, shared and induced
- $q_{\text{edge}} \lesssim 1$

Vertical displacement with M3D-C1

magnetic axis ≡ macroscopic observable ("centre-of-current")
routinely reconstructed from experimental measurements





Vertical displacement model: key points

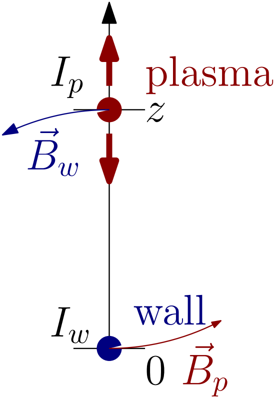
Theoretical study of the non-linear/algebraic trajectory:

- exponential at first, but later always found to decelerate (relaxation) $\Rightarrow$ acceleration indicates sharing of toroidal current (non-inductive effect)
- interpretation/prediction of induced wall currents $\Rightarrow$ initial conditions for modelling subsequent VDE phases
- highlight time-scale separation between Alfvénic and resistive wall dynamics $\Rightarrow$ potential "UV" problem of inductive coupling
- wall structures affect nature of ODEs $\Rightarrow$ numerical representations
  1. 1D: filaments or coils $\rightarrow$ resistance coupling matrix
  2. 2D: two plates or cylindrical shell $\rightarrow$ Laplacian

Reductional assumption

- rigid body approximation of a plasma (lowest order term in Green's function Taylor expansion [Jardin et al., 1986])

Basic VDE model with current carrying wires [Freidberg, 2007]



- plasma is a rigid wire with constant current I_p (free to move vertically only)
- wall is a fixed wire with varying current I_w

$\Rightarrow$ two coupled ODEs (3 first order)

1. plasma acceleration due to Ampère's force

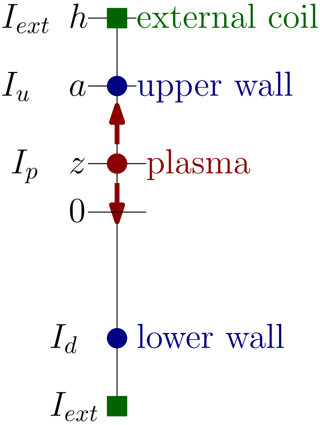
$$\ddot{z} = \int_{\text{plasma}} \frac{dx^3}{m} \boldsymbol{j} \times (\boldsymbol{B}_w + \boldsymbol{B}_{ext}) \propto -\frac{I_p I_w}{z} + f_{ext}$$

2. wall circuit/induction equation

$$L_w \dot{I}_w + R_w I_w = \int_{\text{wall}} d\boldsymbol{l} \cdot (\boldsymbol{v} \times \boldsymbol{B}_p) \propto \frac{I_p \dot{z}}{z}$$

Many wire model and generalisation to Lagrangian

$3 + 2_{ext}$ model



Lagrangian for $z(t)$, $I_p(t)$, $I_u(t)$ and $I_d(t)$

$$\mathcal{L} = \tfrac{1}{2}\dot{z}^2 + \tfrac{1}{2}\lambda_p I_p^2 + \sum_i \left[\tfrac{1}{2}\lambda_i I_i^2 + M(z - z_i) I_p I_i\right] + \sum_{i,j} \tfrac{1}{2} M_{ij} I_i I_j + a_{\mathsf{ext}}(z) I_p$$

Rayleigh dissipation function

$$\mathcal{D} = \frac{1}{2} r_p I_p^2 + \sum_i \frac{1}{2} r_i I_i^2$$

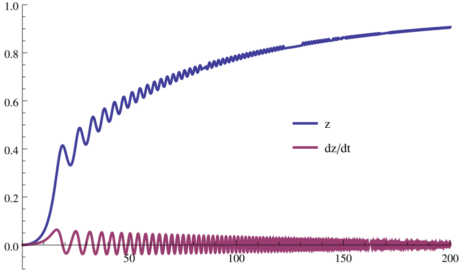
Equations of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} + \frac{\partial \mathcal{D}}{\partial \dot{x}} = 0$$

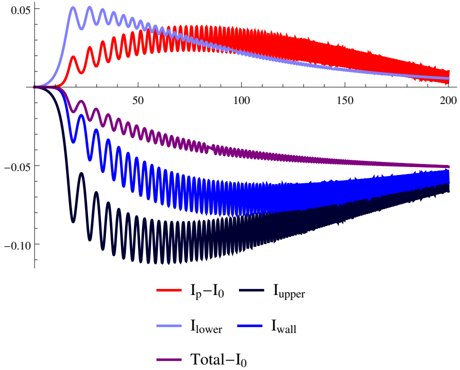
$\Rightarrow$ extension to codes like `VALEN` [Bialek and al.,

Numerical solution of $3 + 2_{ext}$ wire model (Mathematica)

Vertical position and velocity



Plasma current and current in the wall coils during VDE



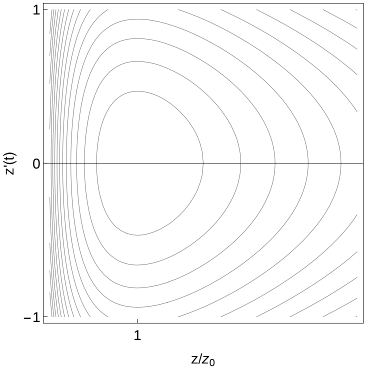
Motion is initially exponential, then oscillatory and finally decelerating

- fast (Alfvén) oscillations are damped + frequency increase (cf. linear theory)
- slow relaxation of average position against the wall is algebraic

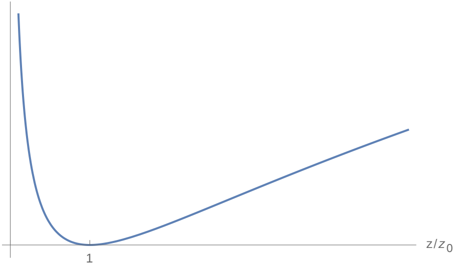
Perfectly conducting wall and effective potential

In the ideal wall limit, $t_A/(L_w/R_w) \to 0$, equations are conservative (integrable)

$$\frac{1}{2}\dot{z}^2 + \underbrace{\frac{2}{L_w}\left(\ln\frac{z}{z_0}\right)^2 + V_{ext}(z)}_{\text{effective potential}} = C$$



phase-space portrait



effective potential $(\ln z/z_0)^2$
(wall at $z = 0$)

$\Rightarrow$ fast (Alfvénic) oscillations around z_0

Role of wall resistivity

- damp the fast oscillations
- allow the plasma to "sink" through the external potential on L_w/R_w timescales (shifting the equilibrium position z_0)

Resistive decay regime:

Neglecting $\ddot{z}$ and $\dot{I}_w \sim O(t_A/(L_w/R_w))^2$

$$\text{(basic model)} \begin{cases} \ddot{z} = 0 = -\frac{2I_w}{z} + f_{ext} \\ \dot{I}_w = 0 = \frac{2\dot{z}}{z} - \frac{R_w}{L_W} I_w \end{cases} \Rightarrow \begin{cases} I_w = \frac{2L_w}{R_w}\frac{\dot{z}}{z} \\ \frac{\dot{z}}{z^2} = \alpha f_{ext} \end{cases}$$

- first-order ODE for $z(t)$ given f_{ext}, independent of inductances
- wall currents prescribed by plasma motion (proportional to displacement rate)

where[1] $\alpha = t_A/S \propto (\mu_0 \sigma a \Delta)^{-1}$

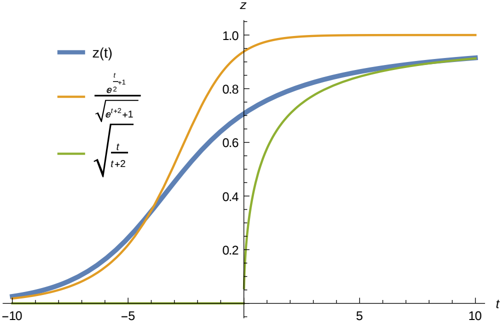
[1] a is distance between wall and initial magnetic axis position (minor radius), Δ is the wall thickness, σ is wall conductivity

Resistive decay regime: non-linear displacement of magnetic axis

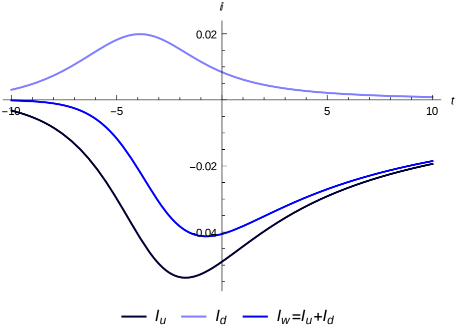
For the $3 + 2_{ext}$ model, in the limit $I_p = I_0$ and $\ddot{z}, \dot{I}_i \ll 1$

$$r_i I_i(t) = \frac{\dot{z}}{z - z_i} \qquad \dot{z}\left[\frac{1}{(1-z)^2} + \frac{1}{(1+z)^2}\right] = \alpha f_{ext}$$

For $f_{ext} = I_p B_H z$, $\ln \frac{z^2}{1-z^2} - 2\frac{1-2z^2}{1-z^2} = I_p B_H \alpha t$

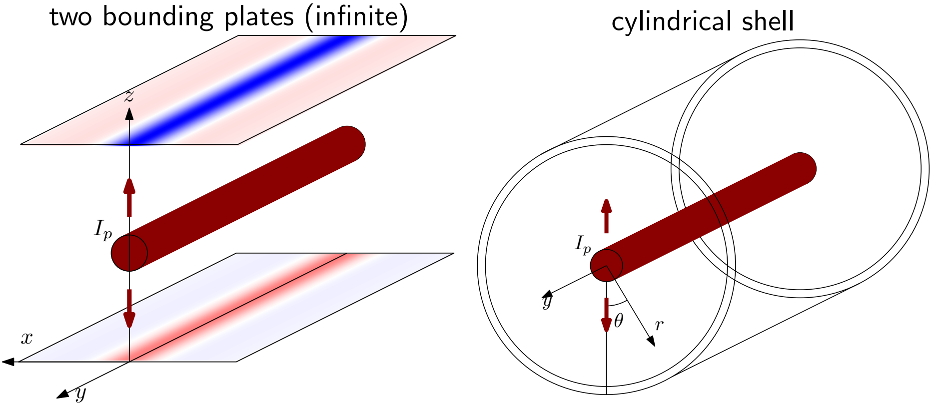


plasma vertical motion
(and asymptotic curves)



induced wall currents

Thin shell model of the wall



- qualitative equivalence between the two models (factor $\pi/2$)
- different dynamics with thin shell than with wires $\rightarrow$ extra dimension allows for additional current diffusion

Nature of differential equations

Maxwell equations ⇒ heat equation[2] for surface currents $\boldsymbol{j} = \boldsymbol{\nabla}\kappa \times \hat{\boldsymbol{n}}$

$$\partial_t \kappa - \frac{1}{\mu_0 \sigma} \boldsymbol{\nabla}^2 \kappa = -\partial_t B_p^z$$

1. perfectly conducting wall ($\sigma \to \infty$): mirror currents

$$\kappa = -B_p^z + \kappa_0$$

2. resistive decay ($\partial_t \kappa \ll 1$): Poisson equation sourced by plasma motion

$$\boldsymbol{\nabla}^2 \kappa = \mu_0 \sigma \dot{z} \partial_z B_p^z$$

[2] as opposed to circuit equation $L_w \dot{I}_w + R_w I_w = V$

Steps to solve cylindrical shell model [Boozer, 2015]

1. Multipole expansion of plasma (point-source) field
$\boldsymbol{B}_p = \boldsymbol{\nabla} \times \boldsymbol{A}_p = -2 \ln r' \to$ harmonic potential $\boldsymbol{B}_p = \boldsymbol{\nabla}\Phi_p$

$$\Phi_p|_{\text{wall}} = \theta + \sum_{m=1}^{\infty} z^m \frac{\sin m\theta}{m}$$

2. Fourier expansion of surface current potential $\boldsymbol{j} = \boldsymbol{\nabla}\kappa \times \hat{\boldsymbol{n}}$

$$\kappa'(\theta, t) = \sum_{m=1}^{\infty} 2I_m(t) \cos m\theta \quad \Rightarrow \quad \Phi_s|_{\text{plasma}} = -\sum_{m=1}^{\infty} I_m z^m \frac{\sin m\theta}{m}$$

3. Faraday + Ohm in wall: circuit equations (diagonal)

$$\frac{d}{dt}\left[I_m + z^m\right] = -2\alpha m I_m$$

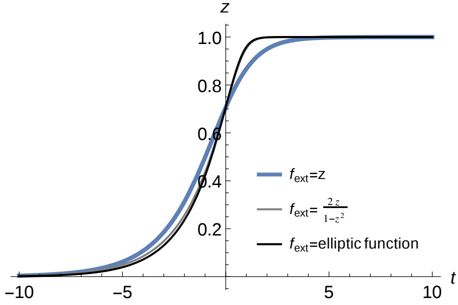
4. Ampère's force on plasma $\ddot{z} = 2\sum_{m=1}^{\infty} I_m(t) z^{m-1} + f_{ext}(z)$

Resistive decay regime: non-linear displacement of magnetic axis

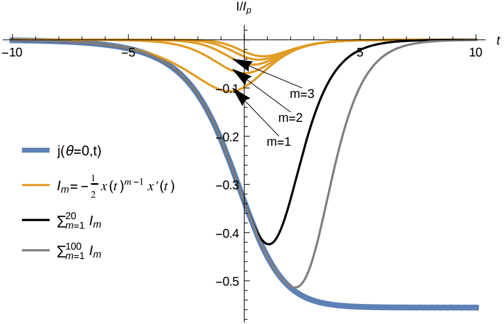
For cylindrical shell (equal result for thin plate), assuming $\dot{I}_m, \ddot{z} \ll 1$,

$$I_m(t) = -\frac{1}{2\alpha}\dot{z} z^{m-1} \qquad \frac{\dot{z}}{2z^2}\sum_{m=1}^{\infty} z^{2m} = \frac{\dot{z}}{1-z^2} = \alpha f_{ext}$$

For $f_{ext} = I_p B_H z$, $\dot{z}\left(\frac{2}{z} + \frac{1}{1-z} - \frac{1}{1+z}\right) = I_p B_H \alpha \Rightarrow z(t) = \frac{1}{\sqrt{1+e^{-I_p B_H \alpha t}}}$



plasma vertical motion



induced currents

Resistive decay regime: non-linear displacement of magnetic axis

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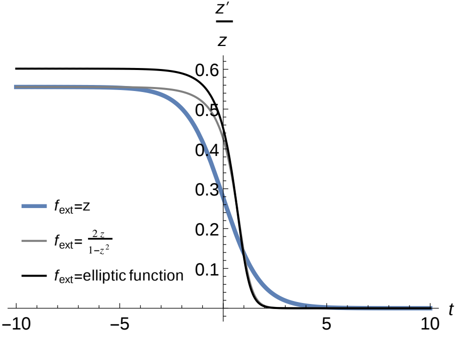
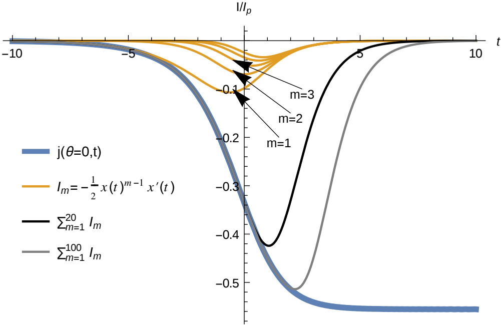


instantaneous growth rate $\dot{z}/z$



induced currents

Summary of results

- analytic model for non-linear vertical motion of plasma (rigid body approximation)
- inductive coupling with wall currents: 1) wires and 2) plates/shell $\rightarrow$ nature of ODEs
- tractable in 1) perfectly conducting wall and 2) resistive decay regime
- induced currents $\rightarrow$ decelerating motion (relaxation)

Conclusions, outlook and future work

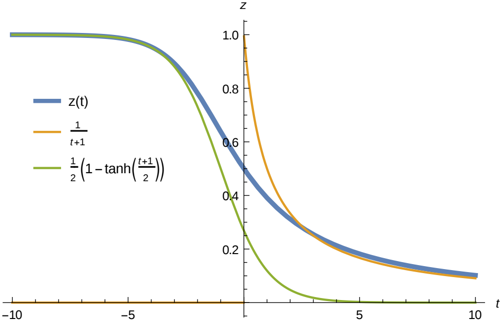
- deviation from behaviour must indicate sharing of toroidal current (early contact of plasma/halo with wall)
  $\Rightarrow$ benchmark simulations + diagnose experimental traces
- extensions of model may provide initial conditions for wall currents to study later phase of VDE
- generalisation of resitive decay regime to 3D $\Rightarrow$ study of non-linear evolution of 3D plasma

Thank you for your attention

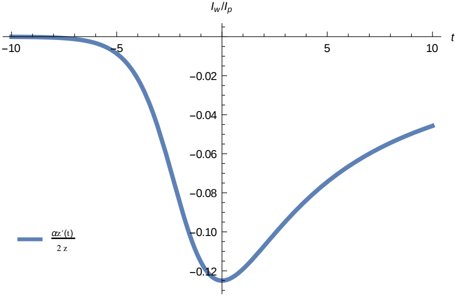
Questions?

Non-linear displacement of equilibrium position (basic case)

For[3] $f_{ext} = I_p B_H (z-1)$, then $\frac{1}{z} - 2[\text{arctanh}(2z-1)+1] = I_p B_H \alpha t$



vertical plasma motion $z(t)$
(and asymptotic curves)



induced wall current I_w/I_p

- algebraic plasma motion is decelerating (relaxation process)
- currents are proportional to displacement rate

[3] where $B_H = \partial_z B_H|_{z=0} \propto I_{ext}$, the vertical variation of horizontal field at $z=0$.

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