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A Lagrangian conservative space and time staggered hydrodynamic scheme

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Introduction, aim

Present trends for Lagrange (and ALE) hydro schemes give preference to **time (and even space) centering** of all conservative quantities

- CAVEAT scheme: [Addessio, Baugardner, Dukowicz, Johnson, Kashiwa, Rauenzahn, Zemach, 1990]
- Compatible-Hydro scheme: [Caramana, Burton, Shashkov, Whalen 1998]
- GLACE scheme: [Després, Mazeran, 2003]
- EUCCLHYD scheme: [Maire, 2004, 2007]

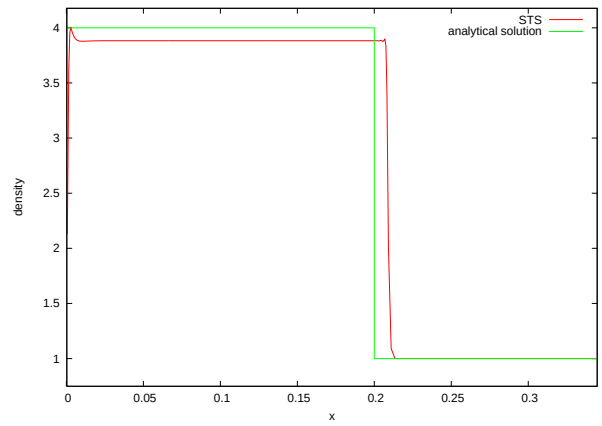
Notations:

"VNR" (Von Neumann–Richtmyer, 1D) $\equiv$ "Wilkins" (2D)
$\equiv$ "SGH" (Staggered Grid Hydro) $\equiv$ "STS" (Space and Time Staggered)

"CSTS" = Conservative Space and Time Staggered

Introduction, aim

Most widely known distortion: shock levels and velocities



Density profiles STS scheme on plane Noh's test:
$\approx$ 5% error on density jump and propagation velocity

However:

- time and space staggered schemes are extremely practical and widely used ([von Neumann–Richtmyer 1950], [Wilkins 1964], [Pracht 1975]. . .)
- yet, little research effort spent on such schemes (hopeless? old fashioned? focus on Q & anti-hourglassing?. . .)
- there is no theorem or proof that conservativity cannot be achieved
- and hints are actually that **total energy can be exactly conserved** ([Trulio & Trigger 1960], **[Burton 1991]**)

Introduction, aim

Here, reexamine, extend, and assess conservative modified STS schemes

I. Write a modified STS scheme with following properties:

- same calculation structure as the STS (1, 2, or 3D)
- second order in time and space as the STS
- exactly conservative in mass, momentum, and total energy
- under all conditions (regardless of time step changes)
- with a kinetic energy defined by a positive definite quadratic form of velocity (rules out Trulio & Trigger's $E_c = \sum_i m_i \mathbf{u}_i^{n+1/2} \mathbf{u}_i^{n-1/2}$)
- and, if possible, second order entropic (bonus, demands energy conservation)

II. Basic tests of impact on:

- energy conservation
- jump conditions
- constraints on time step
- mesh behavior in 2D

Outline

- Continuous system of equations
- "Historical" space and time discretization
- CSTS scheme
  - Conservative Space and Time Discretization
    - Kinetic energy equation
    - Internal energy equation
    - Numerical example: Noh 1D
    - Total energy conservation
    - Rewriting of the internal energy
  - Entropy condition
    - Order one artificial viscosity
    - Order two predicted–corrected artificial viscosity
    - Numerical example: Noh 1D
  - Comments on the effective energy of the system
- Numerical results
  - Sod
  - Noh 2D
  - Kidder

Outline

- Continuous system of equations
- "Historical" space and time discretization

Local continuous system of equations

We solve compressible Euler's equations:

$$
\begin{cases}
\dfrac{d\mathbf{x}}{dt} = \mathbf{u} & \text{semi-Lagrangian configuration} \quad (1a) \\
\dfrac{d\rho}{dt} = -\rho\, \boldsymbol{\nabla} \cdot \mathbf{u} & \text{mass conservation} \quad (1b) \\
\rho \dfrac{d\mathbf{u}}{dt} = -\boldsymbol{\nabla}(p+q) & \text{momentum conservation} \quad (1c) \\
\rho \dfrac{de}{dt} = -(p+q)\, \boldsymbol{\nabla} \cdot \mathbf{u} & \text{internal energy evolution} \quad (1d) \\
p = \mathrm{EOS}(\rho, e) & \text{system closure} \quad (1e)
\end{cases}
$$

where $\mathbf{u}$ is velocity, ρ density, p pressure, q artificial viscosity and e internal energy

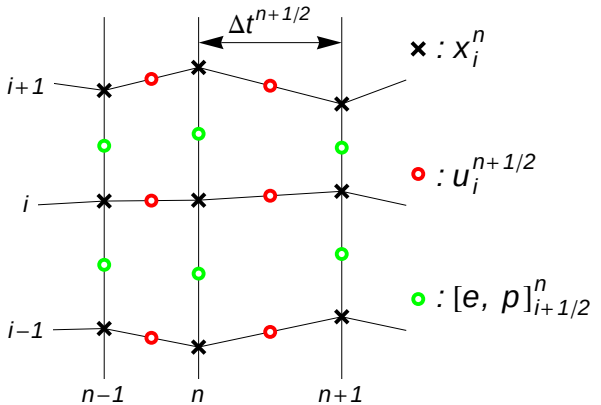
"Historical" space and time discretization

System (1) writes:

$$
\begin{cases}
\mathbf{x}_p^{n+1} = \mathbf{x}_p^n + \mathbf{u}_p^{n+1/2}\Delta t^{n+1/2} & \text{(2a)}\\
m_c^{n+1} = m_c^n & \text{(2b)}\\
m_p(\mathbf{u}_p^{n+1/2} - \mathbf{u}_p^{n-1/2}) = \sum_{c\in C(p)} (p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \frac{\Delta t^{n+1/2}+\Delta t^{n-1/2}}{2} & \text{(2c)}\\
\frac{e_c^{n+1}-e_c^n}{\Delta t^{n+1/2}} = -\frac{(p+q)_c^{n+1/2}}{\Delta t^{n+1/2}}\left(\frac{1}{\rho_c^{n+1}} - \frac{1}{\rho_c^n}\right) & \text{(2d)}\\
p_c^{n+1} = EOS(\rho_c^{n+1}, e_c^{n+1}) & \text{(2e)}
\end{cases}
$$

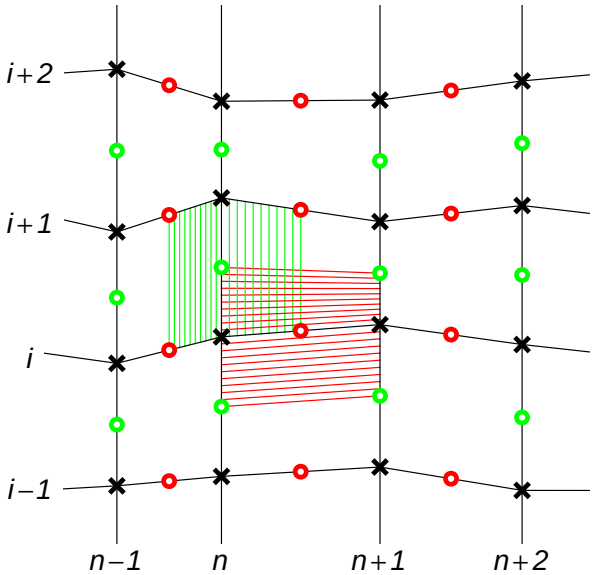
where:

- c cell labels, p node labels
- $\Delta t^{n+1/2}$ time step between t^n and t^{n+1}
- $\frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n$ corner vectors
- $p_c^{n+1/2} = \frac{1}{2}(p_c^n + p_c^{n+1})$
- $m_p = \sum_{c\in C(p)} \frac{1}{|C(p)|} m_c$



Outline

Conservative Space and Time Discretization



Consistency demands that **action** $\mathcal{A}$ (= time integral of energy) be built from kinetic and internal energies discretized over the **STS grid**:

$$E_i^n = \sum_c m_c e_c^n$$

$$E_k^{n+1/2} = \sum_p \tfrac{1}{2} m_p (\mathbf{u}_p^{n+1/2})^2$$

$$\mathcal{A} = \sum_n \tfrac{1}{2}(\Delta t^{n-1/2} + \Delta t^{n+1/2}) E_i^n + \Delta t^{n-1/2} E_k^{n-1/2}$$

- This is a form of "space-time finite volumes approach" (second-order accurate)
- A least action variational principle yields the **only possible** momentum equation: turns out be identical to the original STS scheme
- Now, from there, total energy conservation can be **deduced** so as to be compatible with the discretization of action: turns out there is **only one possible scheme** (for the given formula of E_k)

Conservative Space and Time Discretization

Kinetic energy equation

The **momentum equation**:

$$m_p(\mathbf{u}_p^{n+1/2} - \mathbf{u}_p^{n-1/2}) = \sum_{c \in C(p)} (p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \frac{\Delta t^{n+1/2} + \Delta t^{n-1/2}}{2}$$

multiplied by $\frac{1}{2}\left(\mathbf{u}_p^{n+1/2} + \mathbf{u}_p^{n-1/2}\right)$, we obtain the **kinetic energy equation**:

$$\frac{1}{2} m_p \left[(\mathbf{u}_p^{n+1/2})^2 - (\mathbf{u}_p^{n-1/2})^2\right] = \sum_{c \in C(p)} (p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot \frac{\mathbf{u}_p^{n+1/2} + \mathbf{u}_p^{n-1/2}}{2} \frac{\Delta t^{n+1/2} + \Delta t^{n-1/2}}{2} \quad (3)$$

Conservative Space and Time Discretization

Internal energy equation

We use the same "energy tally" argument as [Burton, 91]:

- internal energy equation must **match** the kinetic energy equation
  $\rightarrow$ **only flux terms are left**
- right hand sides of kinetic and internal energies must be opposite up to both **space and time index rearrangements**

$$
\begin{aligned}
& m_c(e_c^{n+1} - e_c^n) \\
& \quad = \sum_{p \in P(c)} -\frac{1}{2}\left[(p+q)_c^{n+1} \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^{n+1} + (p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n\right] \cdot \mathbf{u}_p^{n+1/2} \Delta t^{n+1/2} \\
& \qquad + \frac{1}{4}(p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot (\mathbf{u}_p^{n+1/2} - \mathbf{u}_p^{n-1/2})\,(\Delta t^{n+1/2} - \Delta t^{n-1/2}) \qquad (4)
\end{aligned}
$$

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$$m_c(e_c^{n+1} - e_c^n) = \sum_{p \in P(c)} -\frac{1}{2}\left[(p+q)_c^{n+1} \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^{n+1} + (p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n\right] \cdot \mathbf{u}_p^{n+1/2} \Delta t^{n+1/2}$$

second order accuracy in time

$$+ \frac{1}{4}(p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot (\mathbf{u}_p^{n+1/2} - \mathbf{u}_p^{n-1/2})\,(\Delta t^{n+1/2} - \Delta t^{n-1/2}) \quad (4)$$

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second order accuracy in time

$$+ \frac{1}{4}(p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot (\mathbf{u}_p^{n+1/2} - \mathbf{u}_p^{n-1/2})\,(\Delta t^{n+1/2} - \Delta t^{n-1/2}) \quad (4)$$

- rearrangement of the remaining terms of (3)
- compatible with causality: no time indices beyond $n+1$
- small and cancels for constant Δt

Conservative Space and Time Discretization

The internal energy equation (4) differs from its version in the STS schemes by 3 important features:

- **artificial viscosity** is now **time centered** as pressure (previously suggested by [Trulio & Trigger, 62])
- the volume variations which produce the pressure work are described by the **scalar products of corner vectors and displacements**: $\frac{\partial V_c}{\partial \mathbf{x}_p} \cdot \mathbf{u}_p$
- a novel **corrective term** is required when the time step fluctuates

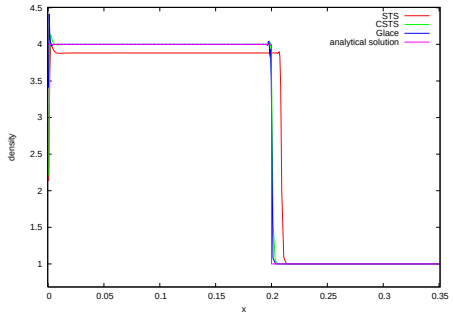
Moreover, there is a minor additional computational time.

Conservative Space and Time Discretization

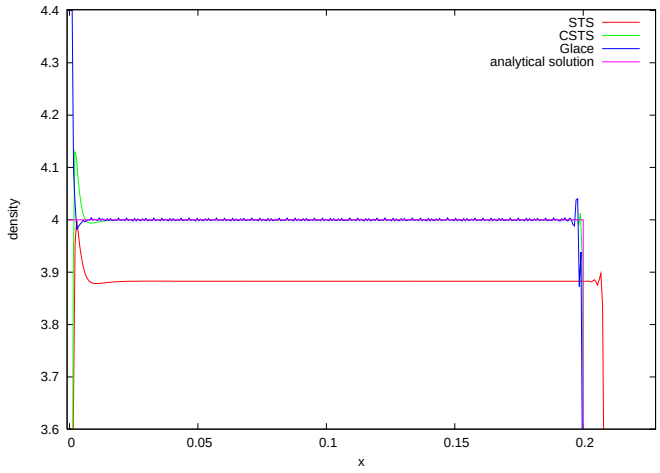
Numerical example: plane 1D Noh's test case

Comparison between: STS, CSTS and cell-centered (GLACE order 2) schemes

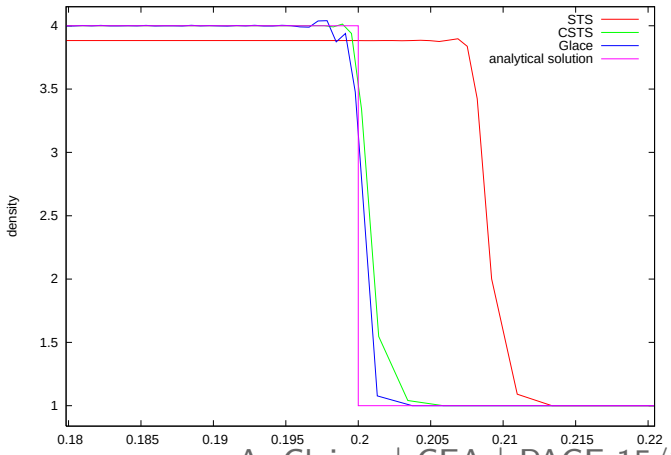
Density profiles at final time $t = 0.6$, CFL$= 0.5$



better shock level



better shock propagation velocity



Conservative Space and Time Discretization

Total energy conservation

Adding (3) and (4) and summing over cells and nodes yields

$$\sum_p \tfrac{1}{2} m_p \left[(\mathbf{u}_p^{n+1/2})^2 - (\mathbf{u}_p^{n-1/2})^2 \right] + \sum_c m_c (e_c^{n+1} - e_c^n)$$

$$= \sum_p \sum_{c \in C(p)} -\tfrac{1}{2} \Big[(p+q)_c^{n+1} \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^{n+1} \cdot \mathbf{u}_p^{n+1/2} \Delta t^{n+1/2}$$

$$- (p+q)_c^{n} \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^{n} \cdot \mathbf{u}_p^{n-1/2} \Delta t^{n-1/2} \Big]$$

where the right hand side appears to be a **flux term in time**

Conservative Space and Time Discretization

Total energy conservation

The conserved numerical energy can thus be written as:

$$E_{\text{Tot.}}^{n-1/2} = E_{\text{Kin.}}^{n-1/2} + E_{\text{Int.}}^{n-1/2} = \sum_p \tfrac{1}{2} m_p (\mathbf{u}_p^{n-1/2})^2 + \sum_c \left[m_c e_c^n + (p+q)_c^n \sum_{p \in P(c)} \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^n \cdot \mathbf{u}_p^{n-1/2} \frac{\Delta t^{n-1/2}}{2} \right]$$

Conservative Space and Time Discretization

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$$+ \sum_c \left[m_c e_c^n + (p+q)_c^n \sum_{p \in P(c)} \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^n \cdot \mathbf{u}_p^{n-1/2} \frac{\Delta t^{n-1/2}}{2} \right]$$

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internal energy backward reconstructed at $t^{n-1/2}$

Conservative Space and Time Discretization

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$$+ \sum_c \left[m_c e_c^n + (p+q)_c^n \sum_{p \in P(c)} \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^n \cdot \mathbf{u}_p^{n-1/2} \frac{\Delta t^{n-1/2}}{2} \right]$$

internal energy backward reconstructed at $t^{n-1/2}$

or

$$E_{\text{Tot.}}^{n-1/2} = E_{\text{Tot.}}^{n} = E_{\text{Kin.}}^{n} + E_{\text{Int.}}^{n} = \sum_c m_c e_c^n$$

$$+ \sum_p \left[\tfrac{1}{2} m_p (\mathbf{u}_p^{n-1/2})^2 + \mathbf{u}_p^{n-1/2} \cdot \sum_{c \in C(p)} (p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^n \frac{\Delta t^{n-1/2}}{2} \right]$$

Conservative Space and Time Discretization

Total energy conservation

The conserved numerical energy can thus be written as:

$$E_{\text{Tot.}}^{n-1/2} = E_{\text{Kin.}}^{n-1/2} + E_{\text{Int.}}^{n-1/2} = \sum_p \frac{1}{2} m_p (\mathbf{u}_p^{n-1/2})^2 \quad \text{positive definite quadratic form}$$

$$+ \sum_c \left[m_c e_c^n + (p+q)_c^n \sum_{p \in P(c)} \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^n \cdot \mathbf{u}_p^{n-1/2} \frac{\Delta t^{n-1/2}}{2} \right]$$

internal energy backward reconstructed at $t^{n-1/2}$

or

$$E_{\text{Tot.}}^{n-1/2} = E_{\text{Tot.}}^{n} = E_{\text{Kin.}}^{n} + E_{\text{Int.}}^{n} = \sum_c m_c e_c^n$$

$$+ \sum_p \left[\frac{1}{2} m_p (\mathbf{u}_p^{n-1/2})^2 + \mathbf{u}_p^{n-1/2} \cdot \sum_{c \in C(p)} (p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p} \Big|^n \frac{\Delta t^{n-1/2}}{2} \right]$$

- non positive definite quadratic form

- kinetic energy forward reconstructed at t^n

Conservative Space and Time Discretization

Total energy conservation

The conserved numerical energy can thus be written as:

$$E_{\text{Tot.}}^{n-1/2} = E_{\text{Kin.}}^{n-1/2} + E_{\text{Int.}}^{n-1/2} = \sum_p \tfrac{1}{2} m_p (\mathbf{u}_p^{n-1/2})^2 \quad \text{positive definite quadratic form}$$

$$+ \sum_c \left[m_c e_c^n + (p+q)_c^n \sum_{p \in P(c)} \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot \mathbf{u}_p^{n-1/2} \frac{\Delta t^{n-1/2}}{2} \right]$$

internal energy backward reconstructed at $t^{n-1/2}$

- internal energy, backward reconstructed to half-integer time indices

$$m_c e_c^{n-1/2} = m_c e_c^n + (p+q)_c^n \sum_{p \in P(c)} \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot \mathbf{u}_p^{n-1/2} \frac{\Delta t^{n-1/2}}{2}$$

- evolution equation of internal energy:

$$m_c (e_c^{n+1/2} - e_c^{n-1/2}) = \sum_{p \in P(c)} -(p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot \frac{(\mathbf{u}_p^{n+1/2} + \mathbf{u}_p^{n-1/2})}{2} \frac{(\Delta t^{n+1/2} + \Delta t^{n-1/2})}{2}$$

Outline

Entropy condition

Order one artificial viscosity

From internal energy evolution equation of $e_c^{n-1/2}$, **entropy condition** $de + pdV > 0$ becomes

$$-q_c^n \sum_{p \in P(c)} \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot \frac{(\mathbf{u}_p^{n+1/2} + \mathbf{u}_p^{n-1/2})}{2} \geq 0 \qquad \text{for any cell } c$$

$\rightarrow$ order two and **implicit**

Order one in time approximation (usual STS schemes):
q_c^n will be an explicit clipped functional of $\{\mathbf{u}_p^{n-1/2}\}$, instead of $\{\frac{1}{2}(\mathbf{u}_p^{n+1/2} + \mathbf{u}_p^{n-1/2})\}$

$$q_c^n = \underline{\mathcal{Q}}_c\left(\{\mathbf{u}_p^{n-1/2}\}\right) = \begin{cases} \mathcal{Q}_c\left(\{\mathbf{u}_p^{n-1/2}\}\right) & \text{if } \mathcal{Q}_c\left(\{\mathbf{u}_p^{n-1/2}\}\right) \sum_{p \in P(c)} \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot \mathbf{u}_p^{n-1/2} < 0 \\ 0 & \text{if } \mathcal{Q}_c\left(\{\mathbf{u}_p^{n-1/2}\}\right) \sum_{p \in P(c)} \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot \mathbf{u}_p^{n-1/2} \geq 0 \end{cases}$$

$\mathcal{Q}_c$ can be any convenient sensible formula (scalar, tensor, TVD, hyper-viscous. . .)

Second order entropy condition

Order two predicted–corrected artificial viscosity

- **no guarantee** that order one explicit q complies with entropy condition
- what about **singularities** and large (variable) Δt?
- significant improvement can be obtained with **predicted–corrected** q which requires **computing the momentum equation twice**:

$$m_p(\mathbf{u}_p^{*n+1/2} - \mathbf{u}_p^{n-1/2}) = \sum_{c \in C(p)} (p_c^n + q_c^{n-1/2}) \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \frac{\Delta t^{n+1/2} + \Delta t^{n-1/2}}{2}$$

$$\text{where} \quad q_c^{n-1/2} = \underline{\mathcal{Q}}_c\left(\{\mathbf{u}_p^{n-1/2}\}\right) \quad \text{predicted} \tag{5a}$$

$$m_p(\mathbf{u}_p^{n+1/2} - \mathbf{u}_p^{n-1/2}) = \sum_{c \in C(p)} (p_c^n + q_c^n) \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \frac{\Delta t^{n+1/2} + \Delta t^{n-1/2}}{2}$$

$$\text{where} \quad q_c^n = \underline{\mathcal{Q}}_c\left(\left\{\tfrac{1}{2}(\mathbf{u}_p^{*n+1/2} + \mathbf{u}_p^{n-1/2})\right\}\right) \quad \text{corrected} \tag{5b}$$

Second order entropy condition

Order two predicted–corrected artificial viscosity

- to preserve energy conservation a correction in the e_c^n equation is required

$$
\begin{aligned}
& m_c(e_c^{n+1} - e_c^n) \\
& = \sum_{p \in P(c)} -\frac{1}{2}\left[(p_c^{n+1} + q_c^{n+1/2})\frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^{n+1} + (p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n\right] \cdot \mathbf{u}_p^{n+1/2} \Delta t^{n+1/2} \\
& \qquad -\frac{1}{2}(q_c^n - q_c^{n-1/2})\frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot \mathbf{u}_p^{n-1/2} \Delta t^{n-1/2} \\
& \qquad + \frac{1}{4}(p+q)_c^n \frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot (\mathbf{u}_p^{n+1/2} - \mathbf{u}_p^{n-1/2})(\Delta t^{n+1/2} - \Delta t^{n-1/2}) \qquad (6)
\end{aligned}
$$

  The e_c^{n+1} expression now does not involve q_c^{n+1} (replaced by available $q_c^{n+1/2}$).

- in this case, $e_c^{n-1/2}$ equation is not modified since this equation uses only corrected q

$$
m_c(e_c^{n+1/2} - e_c^{n-1/2}) = \sum_{p \in P(c)} -(p_c^n + q_c^n)\frac{\partial V_c}{\partial \mathbf{x}_p}\Big|^n \cdot \frac{(\mathbf{u}_p^{n+1/2} + \mathbf{u}_p^{n-1/2})}{2} \frac{(\Delta t^{n+1/2} + \Delta t^{n-1/2})}{2}
$$

CSTS scheme

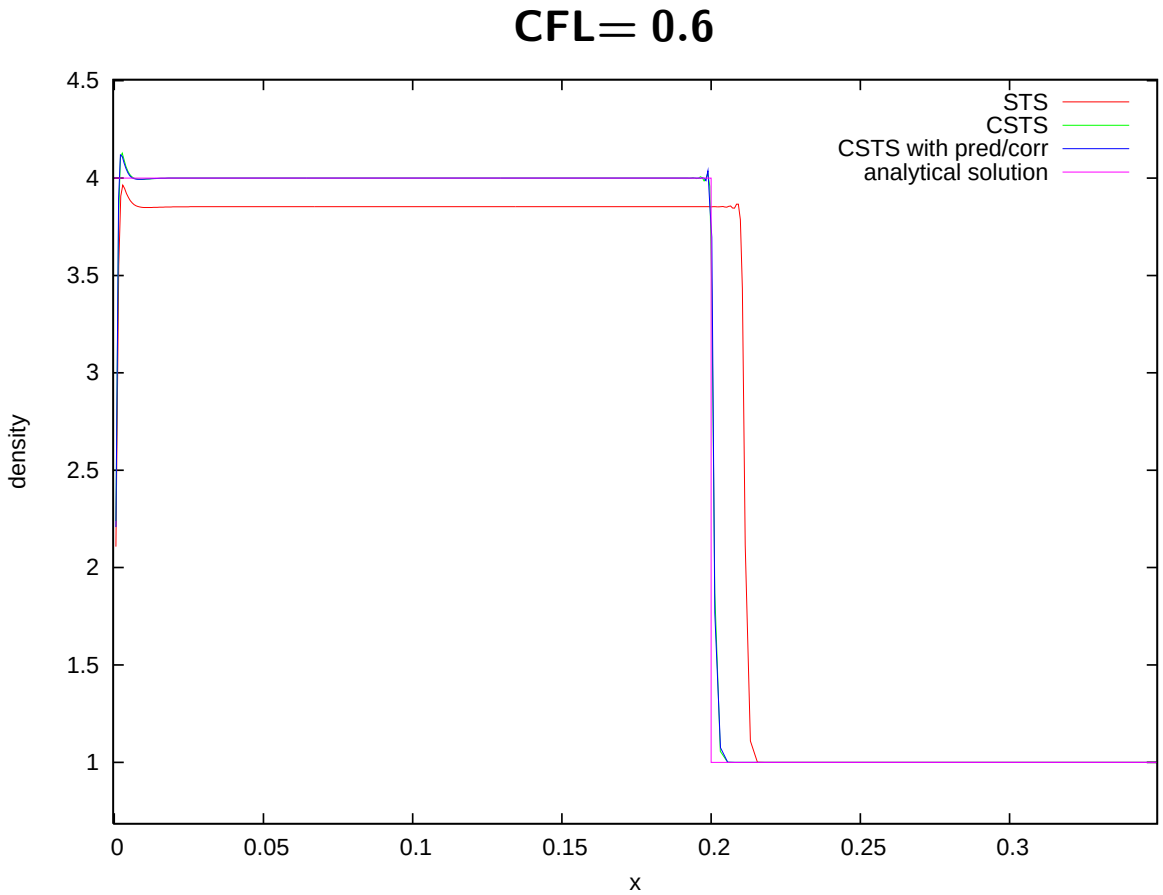
This **CSTS scheme** (5) and (6) (with predicted-corrected artificial viscosity) is now:

- **fully conservative** in momentum and total energy
- **second-order accurate** in entropy production, and
- retains the **locally implicit structure** of original STS schemes (VNR or Wilkins)

Second order entropy condition

Numerical example: plane 1D Noh's test case

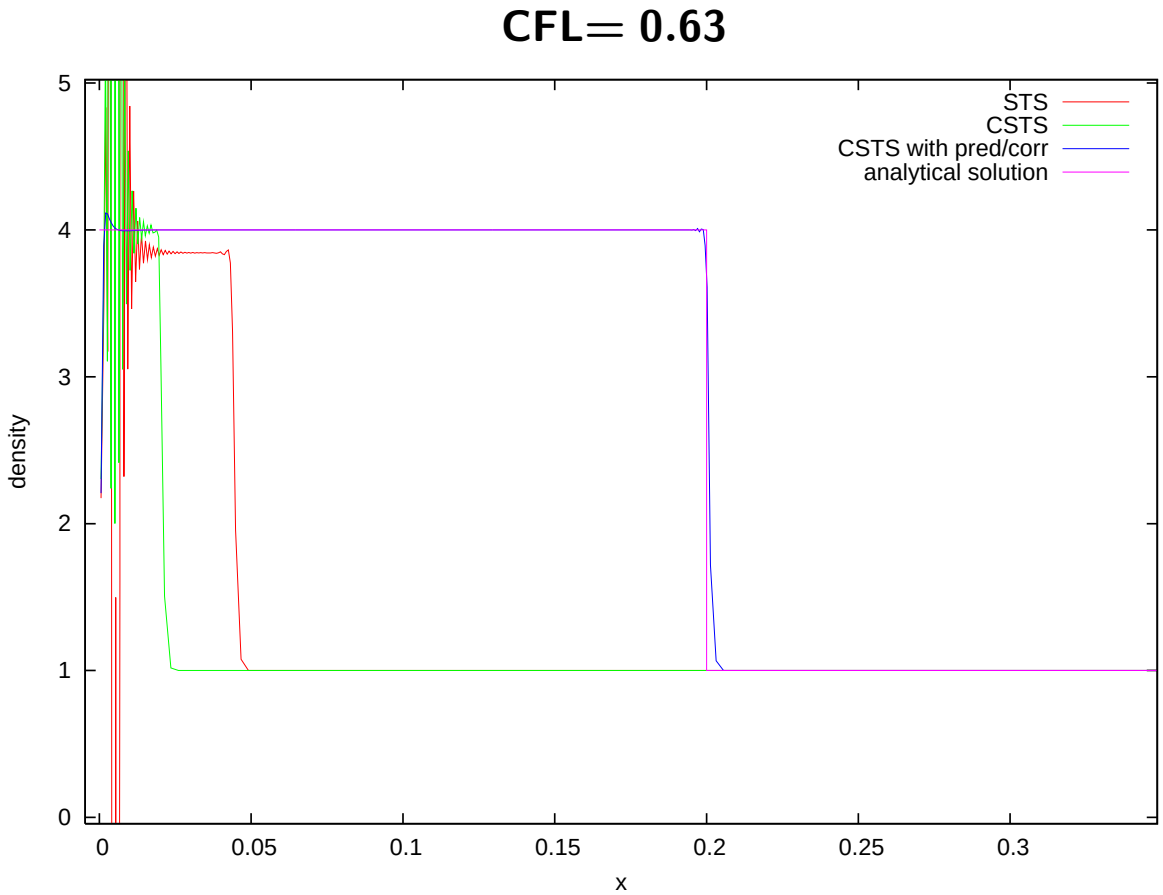
Influence of CFL coefficient on STS and CSTS (without and with pred-corr q):



Second order entropy condition

Numerical example: plane 1D Noh's test case

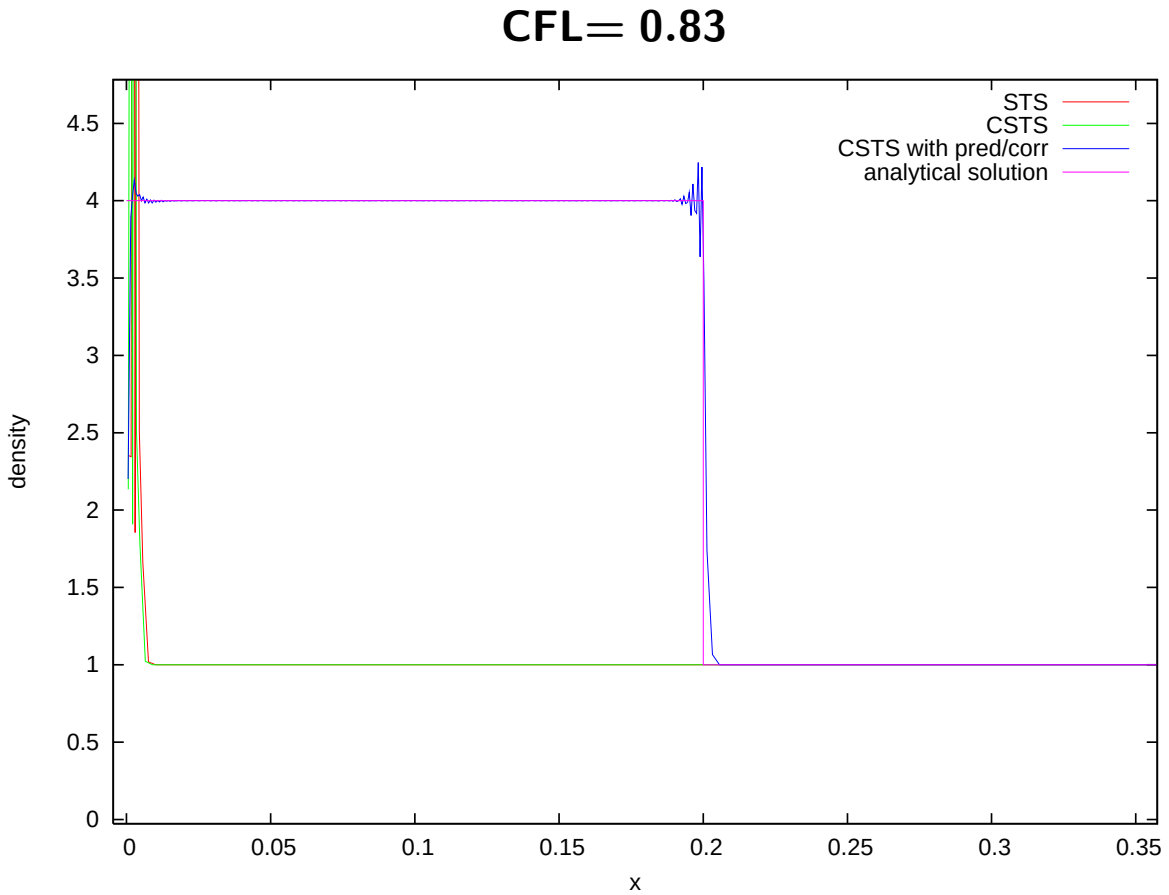
Influence of CFL coefficient on STS and CSTS (without and with pred-corr q):



Second order entropy condition

Numerical example: plane 1D Noh's test case

Influence of CFL coefficient on STS and CSTS (without and with pred-corr q):

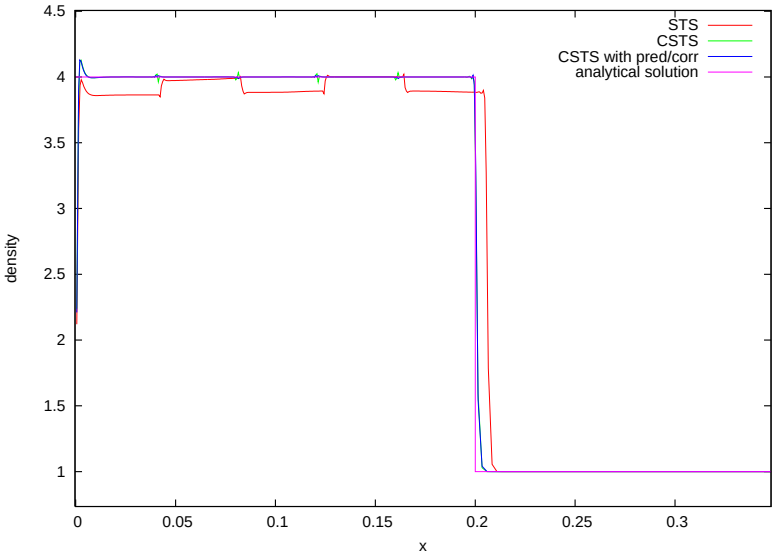


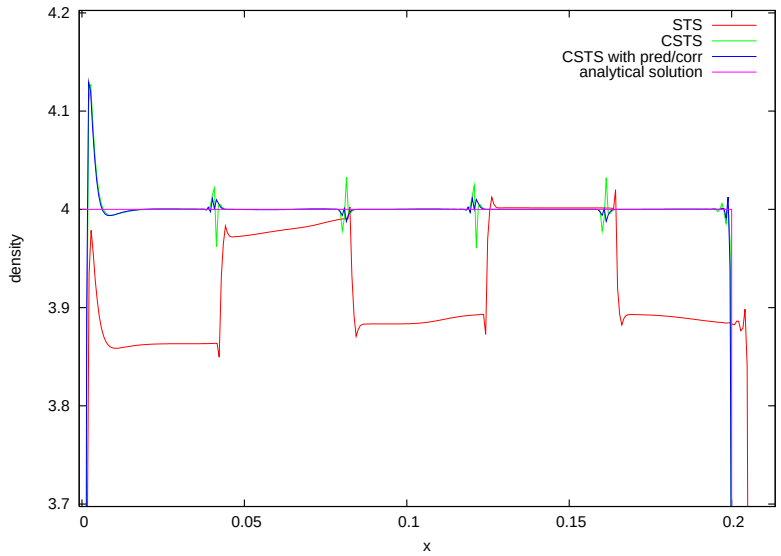
Second order entropy condition

Numerical example: plane 1D Noh's test case

Influence of CFL coefficient on STS and CSTS (without and with pred-corr q):

Variable CFL (0.5 and 0.1)





Outline

Comments on the effective energy

e_c^n on one hand:

- is effectively calculated by the scheme
- is used for pressure calculation (EOS)
- is not associated with a positive quadratic form of kinetic energy
- does not enforce entropic conditions

$e_c^{n-1/2}$ on the other hand:

- is not calculated by the scheme
- is not used for pressure calculation (EOS)
- is associated with a positive quadratic form of kinetic energy
- does enforce entropic conditions

Both definitions are consistent up to the accuracy order of the scheme:

$$e = e_c^n + \mathcal{O}(\Delta x^2) = e_c^{n-1/2} + \mathcal{O}(\Delta x^2) \text{ at given CFL condition}$$

Outline

Numerical results

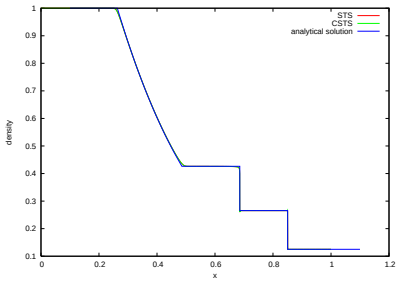
Generalities

- From now on, all results for CSTS scheme use **predicted–corrected** artificial viscosity
- **Linear** artificial viscosity active in both expansion and compression: $q_1 = 0.5$
- **Quadratic** artificial viscosity active in compression only: $q_2 = \frac{\gamma+1}{4}$

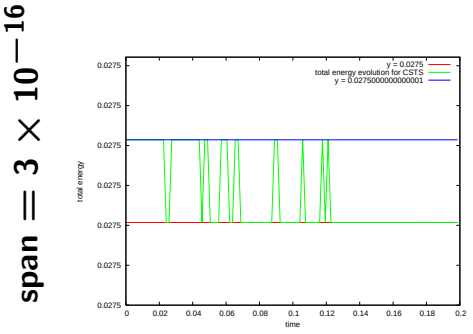
Numerical results

SOD's test case: mesh $[5000 \times 2]$

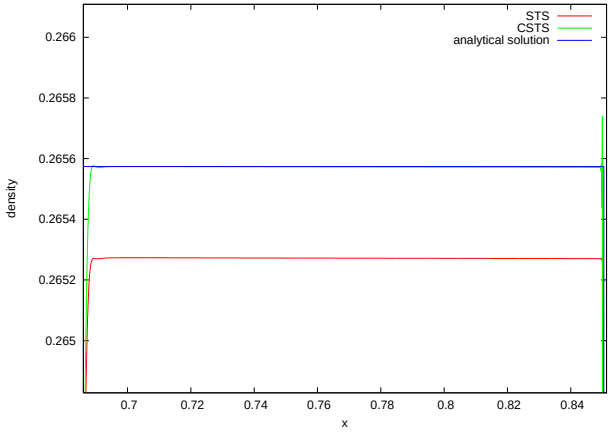
Density profiles at final time $t = 0.2$, CFL $= 0.25$ (STS and CSTS schemes)



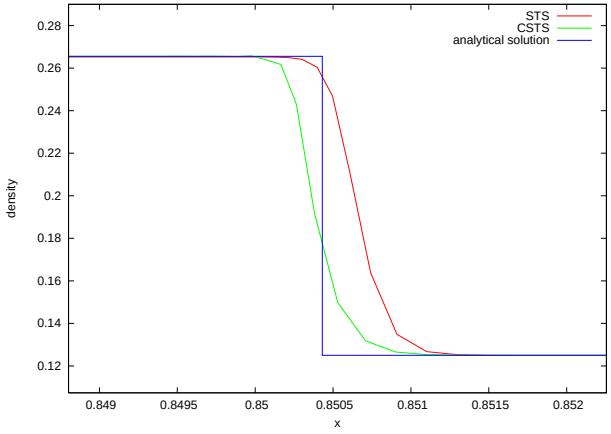
Evolution of total energy for CSTS scheme



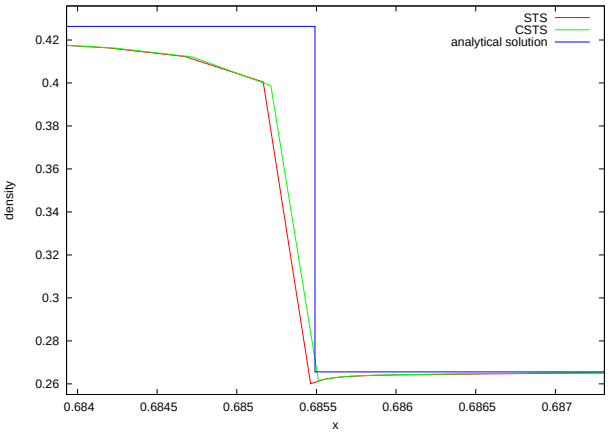
Post shock region



Shock profile



Profile of the contact discontinuity

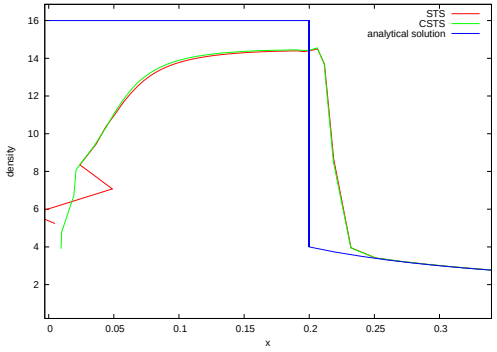


Numerical results

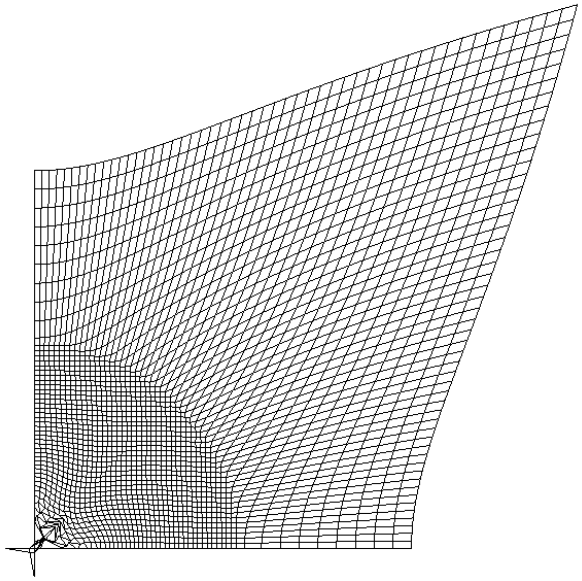
Noh's test case: 2D cylindrical

- cartesian mesh
- $\rho = 1$, $p = 10^{-15}$, $\gamma = \frac{5}{3}$
- CFL$= 0.1$
- no anti-hourglassing algorithm
- q's length scale $\equiv$ width of inertia ellipsoid along radius

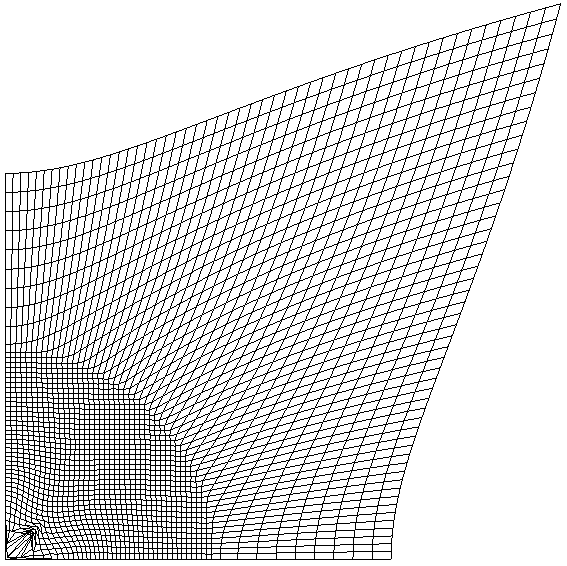
Density profiles at final time $t = 0.6$



Final mesh for STS scheme



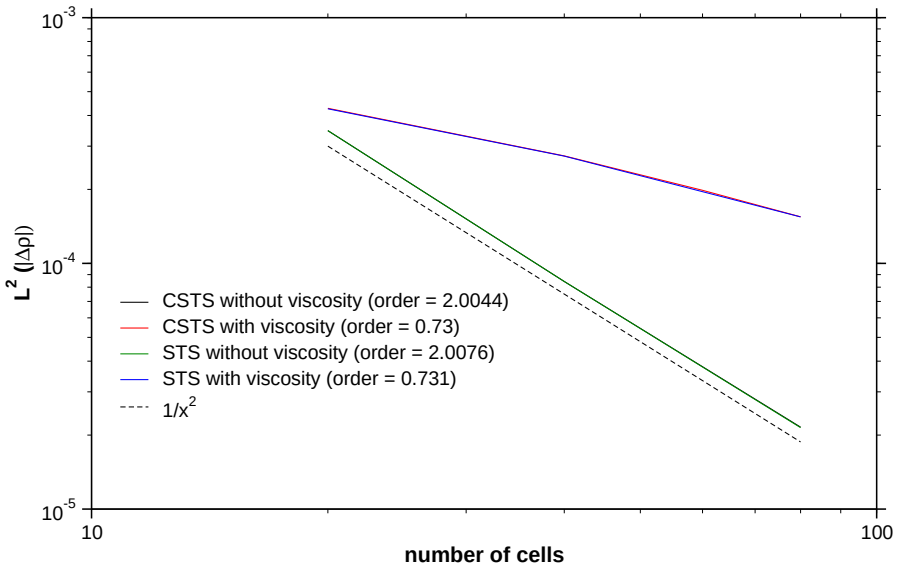
Final mesh for CSTS scheme



Numerical results

Kidder's test case

Computation of the scheme's order on the density profile



3 meshes:
$[20 \times 20]$, $[40 \times 40]$, $[80 \times 80]$

$$E_{L_2}(\rho) = \sqrt{\sum_c \Delta\rho^2}$$

$$\Delta\rho = (\rho_{theo} - \rho_{exp})$$

Conclusion and perspectives

Conclusion

The CSTS scheme is:

- **conservative** for total energy
- **only possible** extension of usual STS schemes
- **second order** for the entropy condition

Simple numerical tests show:

- **better shock capture** (level and propagation)
- **improved 2D robustness**
- **improved CFL margin**
- reduced additional computational time: here 1% and 28% (without and with pred/corr, no Newton on EOS)

Current works

- to be submitted soon
- stability (CFL) analysis
- tests on complex EOS (shock separations. . .)
- other physics and numerics: ALE, 2D axisymmetric, elasticity, chemical reaction. . .

Acknowledgments

P. Le Tallec