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Toward a Reduction of Mesh Imprinting

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Goal: Contribution to Lagrangian Hydro

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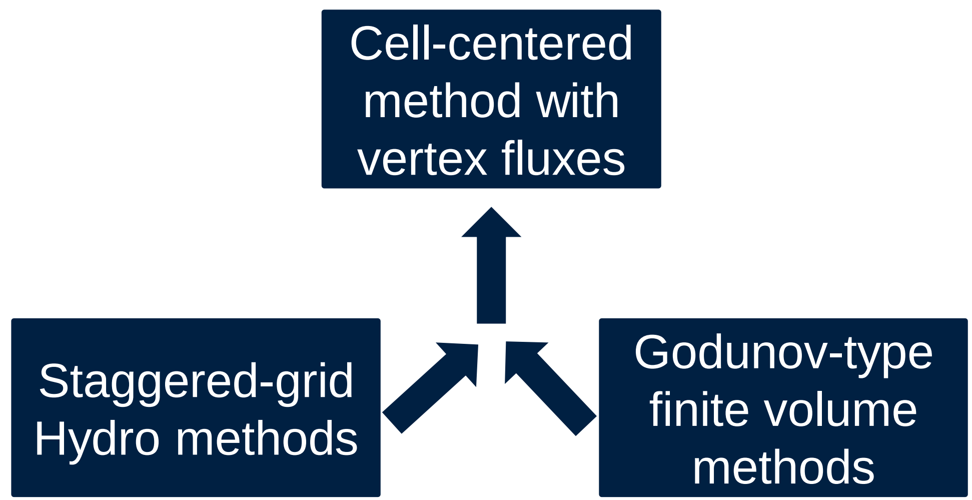
Many researchers have shown that cell-centered hydrodynamic algorithms can be successful in addressing problems associated with the staggered-grid approach

Areas of continuing interest include, **nodal movement**, **spurious vorticity**, **symmetry preservation**, and **mesh imprinting**

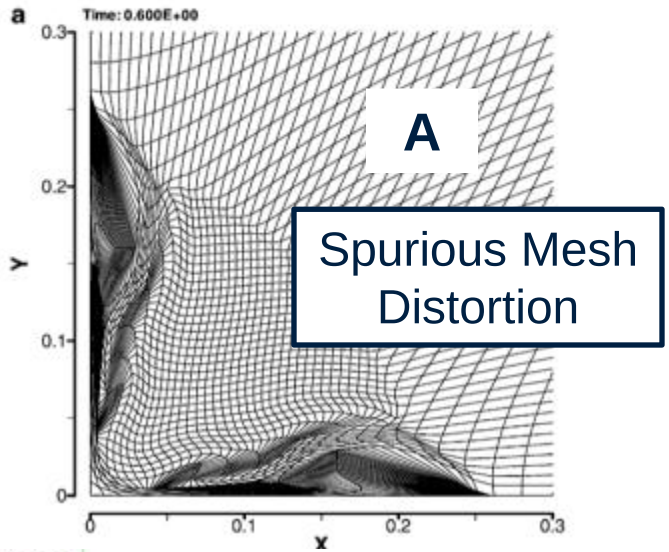
<u>This work aims to address these issues via a new cell-centered approach</u>

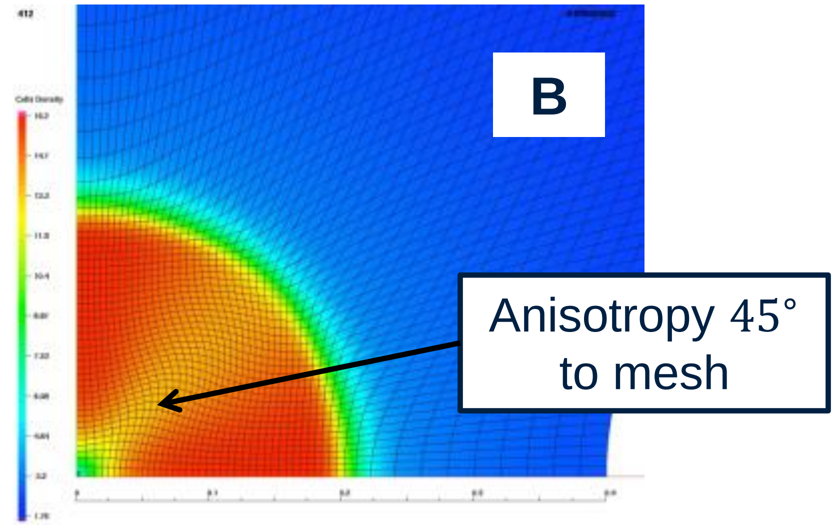
Specific objectives include:

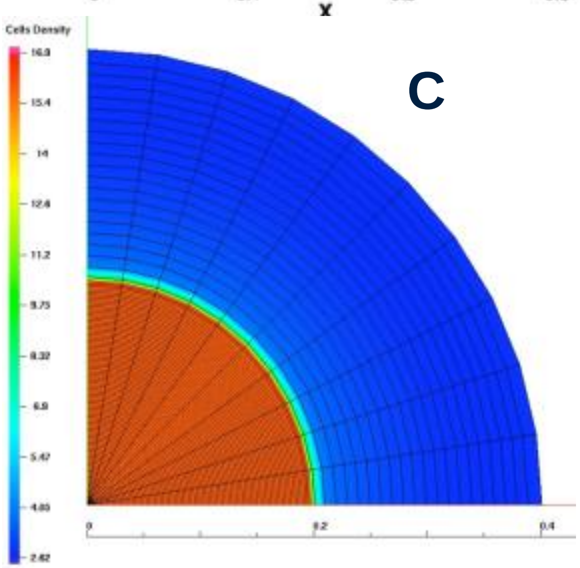
1. Construction of a multi-dimensional algorithm
2. Automatic consistency of mesh motion and fluxes
3. Implementation of affordable vorticity control
4. A “clean” algorithm with minimal complexity



Mesh Imprinting and Tangling







<u>Noh Problem:</u>

A: Staggered-grid Hydro, Cartesian mesh

B: Cell-centered Hydro, Cartesian mesh

C: Cell-centered Hydro, radial mesh

Images: D. E. Burton, et. al. Los Alamos National Laboratory. LA-UR-09-03132

Research Strategy

1. Identify a simplified test environment

<u>2D Acoustics</u>

- Linear physics
- Square mesh
- Intrinsically multidimensional

$$p_t + \rho_0 {a_0}^2 (u_x + v_y) = 0$$
$$u_t + {\rho_0}^{-1} p_x = 0$$
$$v_t + {\rho_0}^{-1} p_y = 0$$

2. Use the following tools to address problem areas
   a) Vorticity control
   b) Dispersion analysis
   c) Nonlinear limiters
   d) Increased order of accuracy

(a), b) → Crisis 1: Mesh Imprinting; c) → Crisis 2: Overshoots)

3. Extend lessons learned to the full problem

Tool: Vorticity Control

Dukowicz and Meltz [1] implemented vorticity control using a costly first order procedure for removing vorticity

- Effective in solving the Saltzman problem

Morton and Roe [2] pointed out that

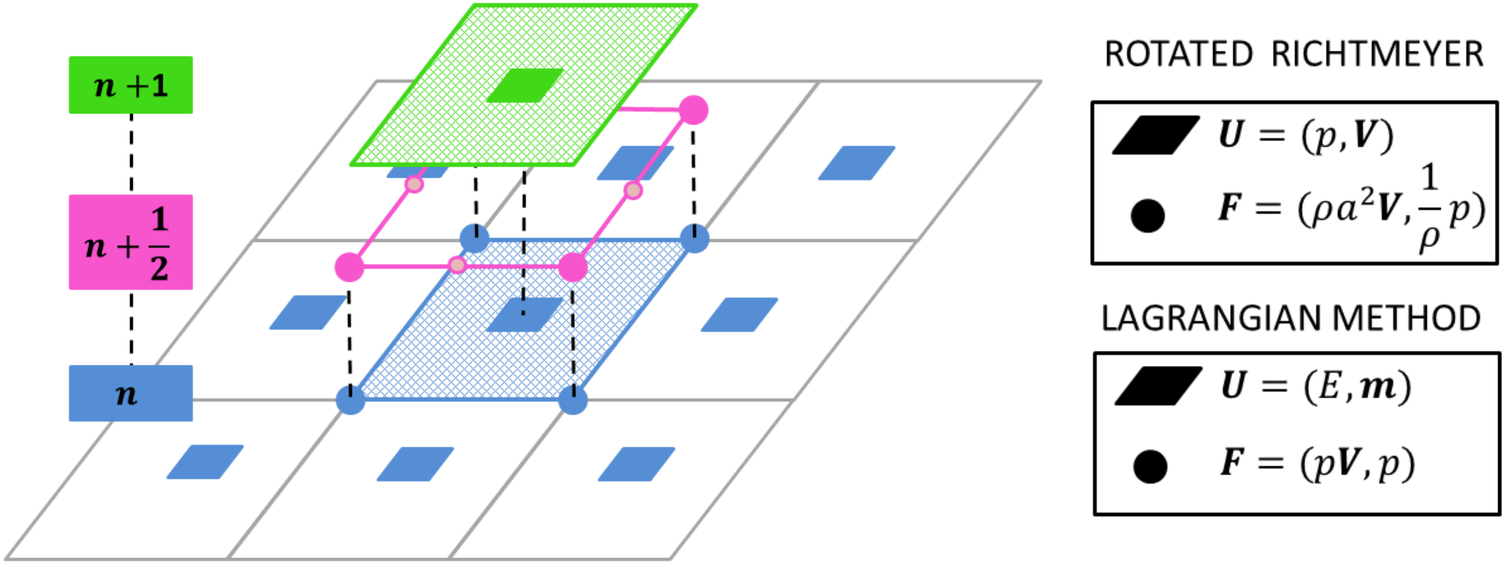
- the Rotated Richtmeyer (RR) scheme, a Lax-Wendroff (LW) variant, creates no spurious vorticity
- vorticity preservation is not attainable using schemes based on one dimensional physics
- fluxes must be calculated at vertices and averaged over faces

Additionally, we propose a nonlinear limiter that retains vorticity preservation

Could the RR scheme, with a limiter, form the basis for a successful Lagrangian hydro scheme?

A Lagrangian Friendly Structure

When used to solve the acoustic equations, the RR scheme can be interpreted as a <u>linearized Lagrangian method</u>

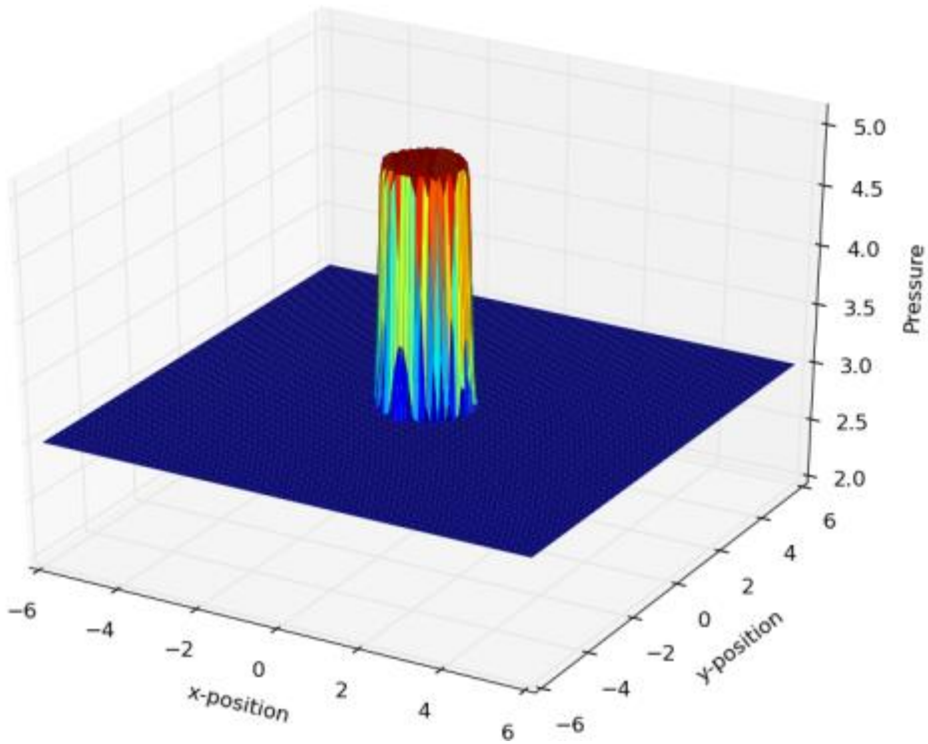


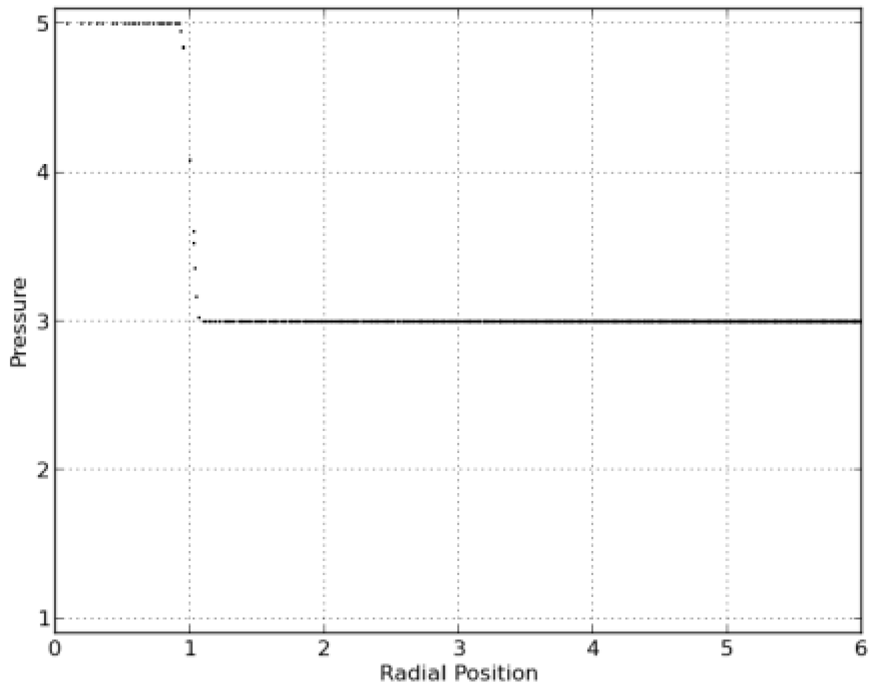
The full Lagrangian version would look much the same:

1. Solve the Eulerian equations on grid that moves with the fluid
    a) Calculate nodal fluxes at $n+\frac{1}{2}$, leaving $p, p\boldsymbol{V}$ stored at vertices
    b) Move the mesh
    c) Update cells using Trapezium Rule
2. Momentum and total energy are conserved
3. A discrete Kelvin Theorem is obeyed on the distorting grid

Test Problem

Discontinuous pressure disturbance introduced to a fluid at rest

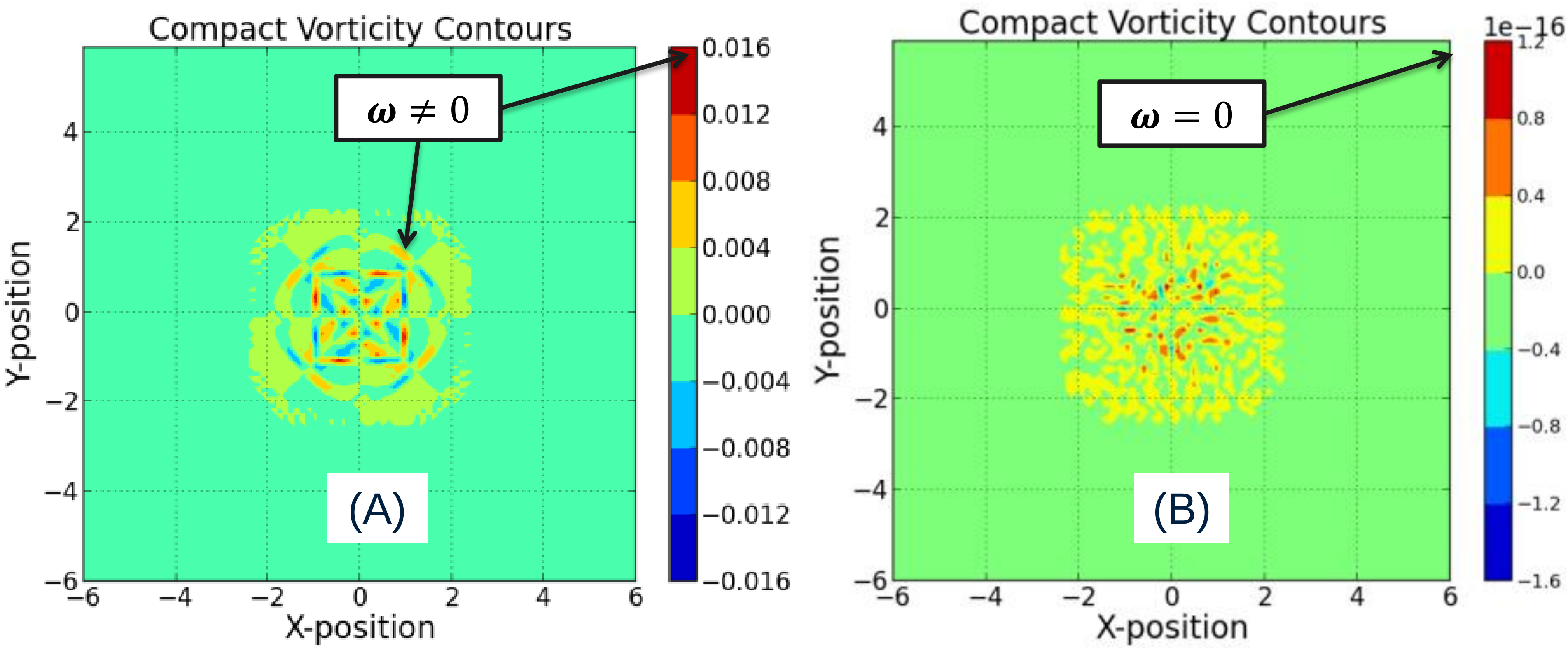




Notes:

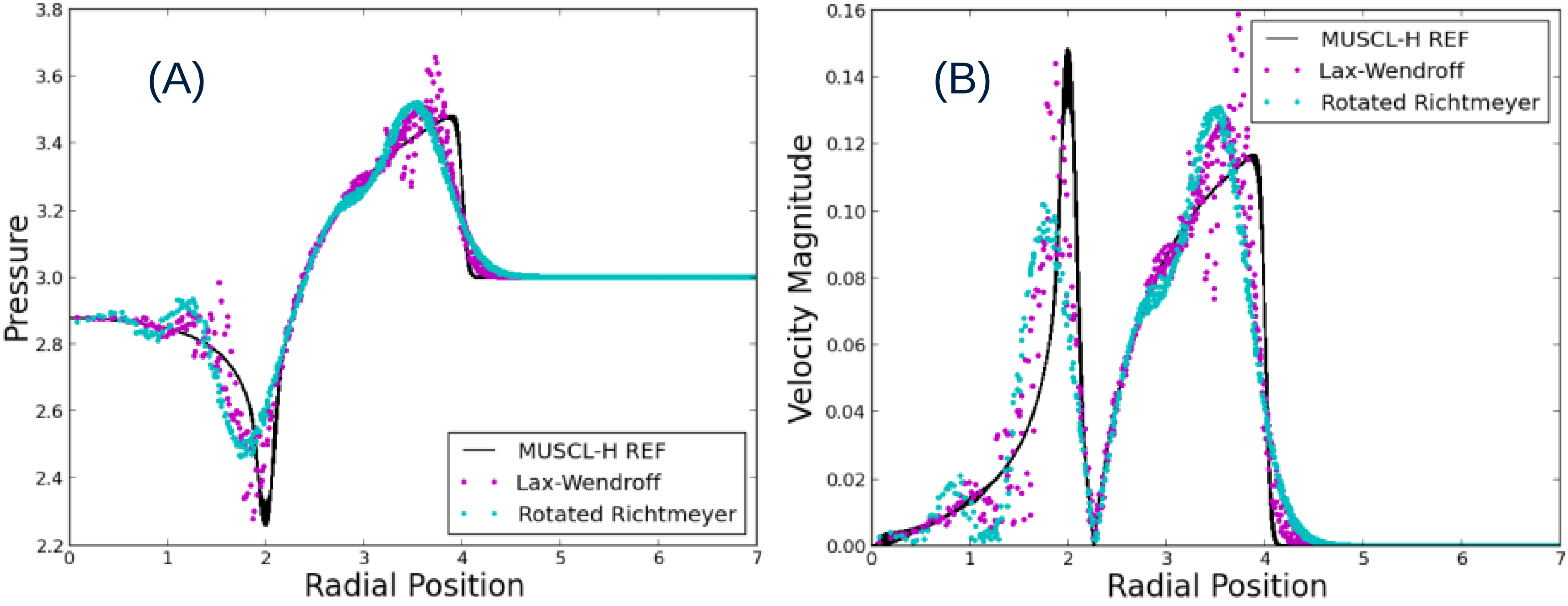
1. All test problems were computed on a 100x100 square mesh unless otherwise noted
2. A reference solution computed using MUSCL-H on a 600x600 mesh is included in most plots

Lax-Wendroff and Rotated Richtmeyer



Compact Vorticity after 10 Time Steps, $\nu = 0.7$:
(A) LW (B) RR

Lax-Wendroff and Rotated Richtmeyer



RR Improvement over LW, $\nu = 0.7$:
(A) Pressure (B) Velocity Magnitude

Improvements to Rotated Richtmeyer

Write the general form of the RR scheme as

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + \underline{M}\boldsymbol{U}^n \qquad \boldsymbol{U} = (p, u, v)$$

where

$$\underline{M} = \begin{pmatrix} -\frac{\nu^2}{2}(\mu_y^2\delta_x^2 + \mu_x^2\delta_y^2) & \nu\mu_x\mu_y^2\delta_x & \nu\mu_x^2\mu_y\delta_y \\ \nu\mu_x\mu_y^2\delta_x & -\frac{\nu^2}{2}\mu_y^2\delta_x^2 & -\frac{\nu^2}{2}\mu_x\mu_y\delta_x\delta_y \\ \nu\mu_x^2\mu_y\delta_y & -\frac{\nu^2}{2}\mu_x\mu_y\delta_x\delta_y & -\frac{\nu^2}{2}\mu_x^2\delta_y^2 \end{pmatrix}$$

Modifications are possible

Modifications not compatible with vorticity preservation

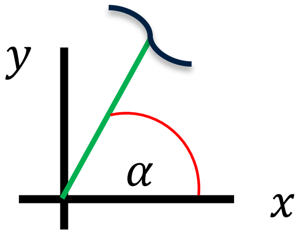
Two free parameters remain

Tool: Dispersion Analysis

Write the parameterized scheme in the form

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^{n} + \underline{T}\boldsymbol{U}^{n}$$

and then carry out a 2D von Neumann substitution



Assume solution with a plane wave propagating in any direction

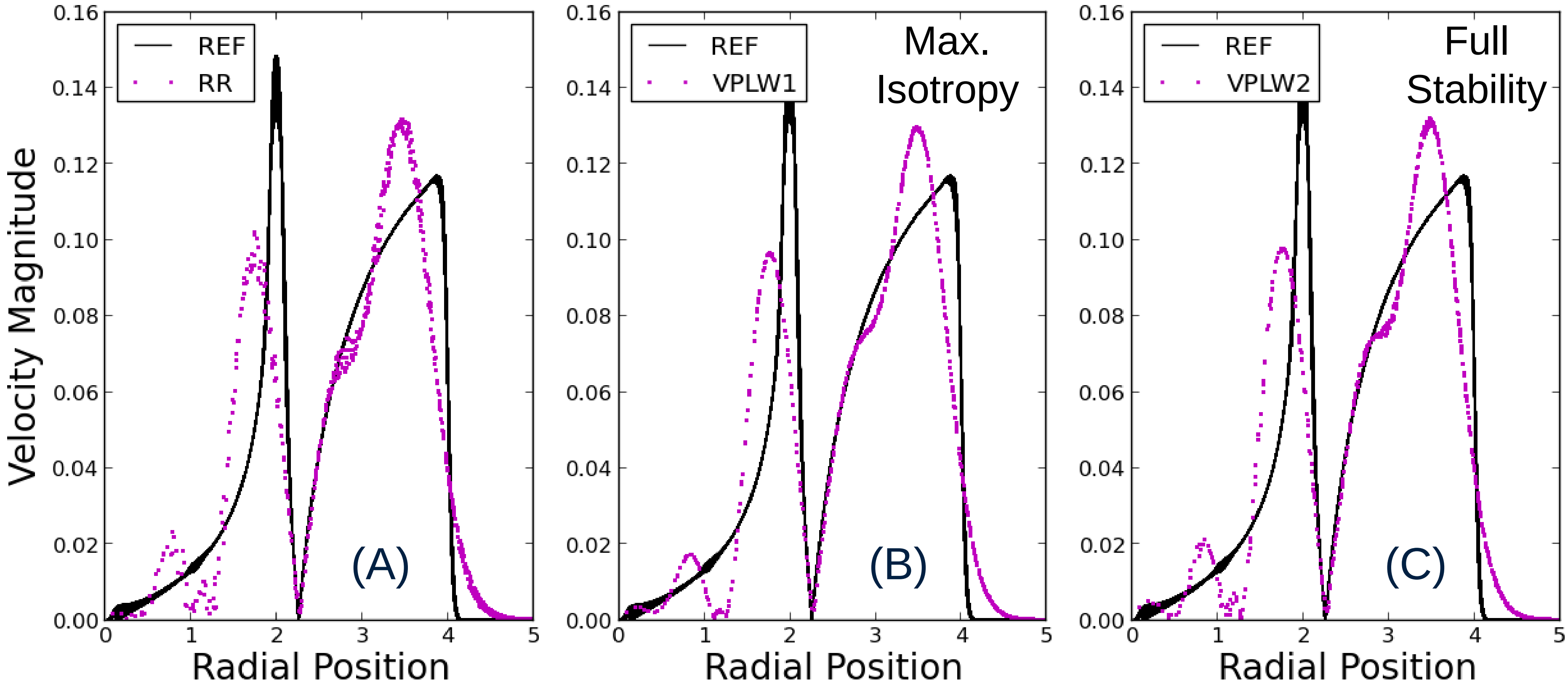
The standard eigenvalue problem can now be recovered and the eigenvalues, g, computed

$$\hat{\underline{T}}\boldsymbol{r} = g\boldsymbol{r}$$

$\mathrm{Abs}(g)$ gives amplification factors
$\mathrm{Arg}(g)$ gives phase change

Expand the eigenvalues in terms of θ_r and pick free parameters that minimize the dependence of the numerical dispersion relations on α

New Vorticity Preserving Schemes



Increased Isotropy of New Vorticity Preserving Schemes, $\nu = 0.6$:
(A) RR No Limiter (B) VPLW1 No Limiter (C) VPLW2 No Limiter

VPLW2 has improved isotropy and maximal stability – write in finite volume form (VPFV2) and try to eliminate overshoots with flux limiting

Crisis 2: Spurious Overshoots

Tool: Nonlinear Limiter

Need a limiting mechanism that is **multidimensional**, **universal**, and **"intelligent"**

$$p_t + \rho_0 {a_0}^2 \nabla \cdot \boldsymbol{V} = 0$$

$$\boldsymbol{V}_t + {\rho_0}^{-1} \nabla p = 0$$

Two fundamental questions:

1. What quantities should be limited?
   a. Conserved variables
   b. Primitive variables
   c. Characteristic variables
   d. Driver quantities $\stackrel{\text{def}}{=} \beta$
      - Pressure Equation: $\beta = \nabla \cdot \boldsymbol{V}$
      - Velocity Equation: $\beta = |\nabla p|$
2. How do you define "monotonicity" in greater than one spatial dimension or with nonlinear physics? (i.e. How much to limit?)
   - Take inspiration from Flux-corrected Transport (FCT) and use **"cautious" first order solution**

Choosing a First Order Scheme

What do we mean by a "cautious" first order scheme?

- Ideal method would have minimum diffusion needed to prevent spurious extrema, introduce minimal phase error, preserve vorticity, and be isotropic

Consider the 1D Q-schemes for linear advection that use a three point stencil:

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{q}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

Optimal Diffusion: First Order Upwind (FUP), $q = |\nu|$

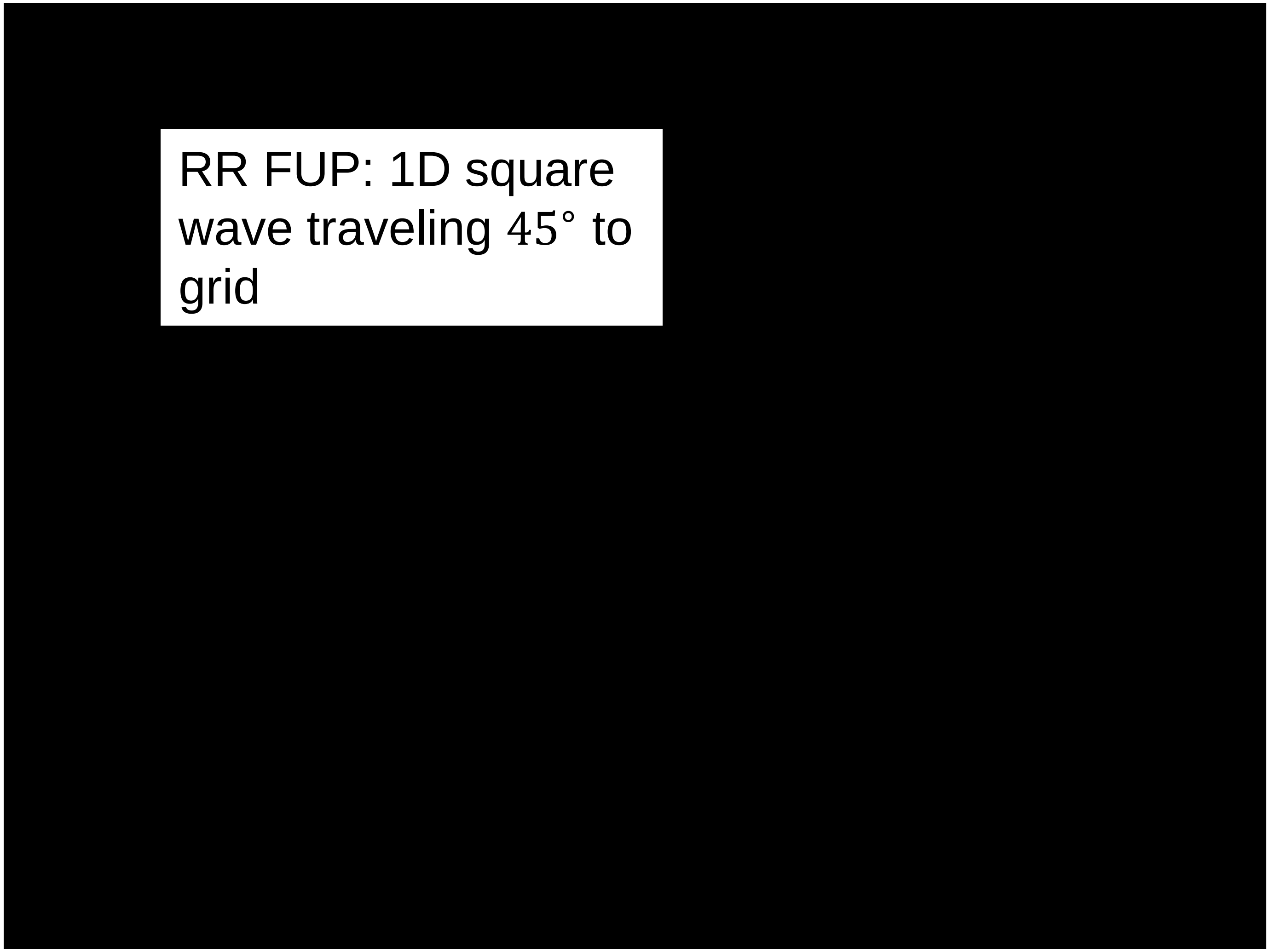
Optimal Phase: Low Phase Error Scheme (LPE), $q = \frac{1+2\nu^2}{3}$

Second Order : Lax-Wendroff Scheme (LW), $q = \nu^2$

Consider the 2D analog of the Q-schemes for the acoustic system:

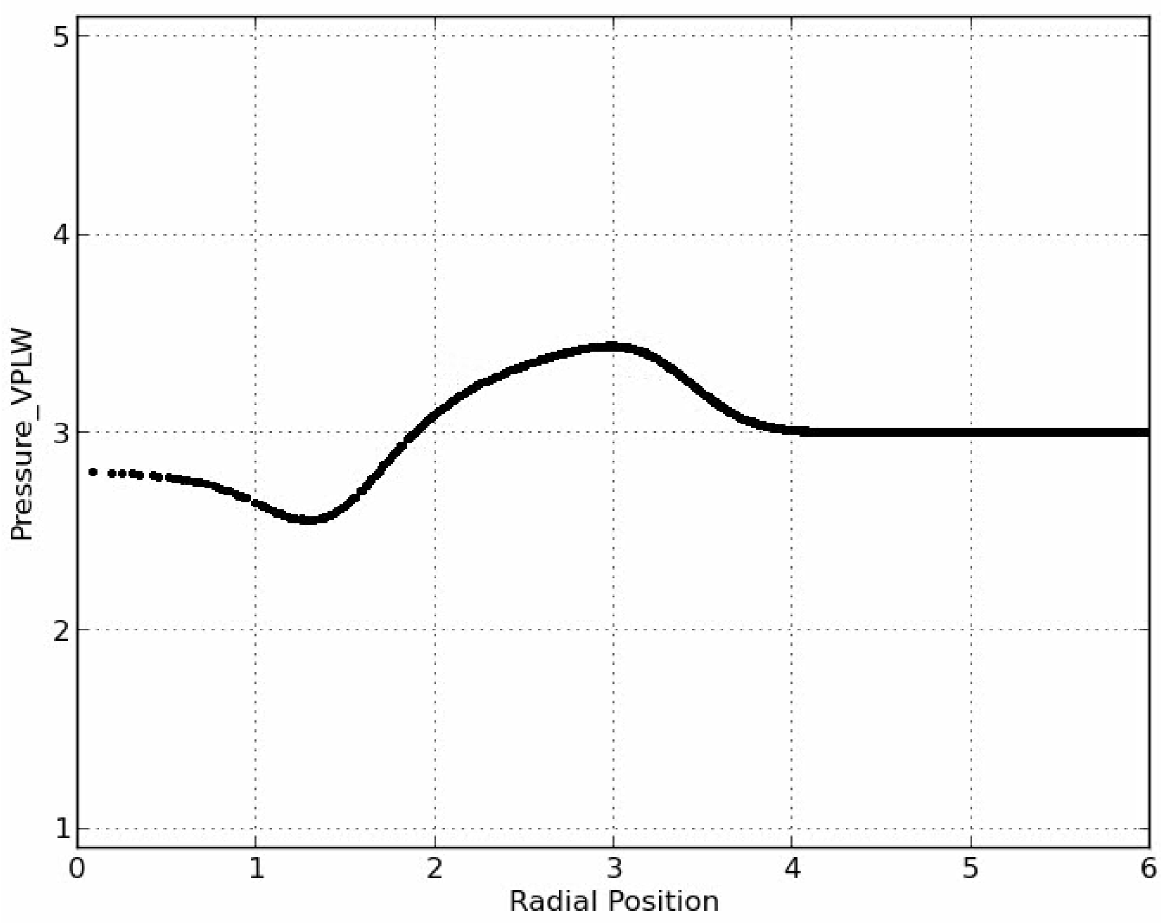
$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + \nu \underline{M^1}\, \boldsymbol{U}^n + q \underline{M^2} \boldsymbol{U}^n$$

Choosing a First Order Scheme



Choosing a First Order Scheme

Best results to date obtained with VPLW2 weights and $q = 0.8\nu + 0.2\nu^2$



How can we incorporate this first order scheme into a useful limiter?

Starting Point: FCT

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FCT in Brief (Boris and Book [3])

1. Compute cautious first order step
2. Compute antidiffusive fluxes (defined using a higher order method)
3. Correct the antidiffusive fluxes using a nonlinear limiter
4. Compute final update with the limited antidiffusive fluxes to remove as much diffusion as possible

Original flux limiter was derived for one dimension and was prone to clipping

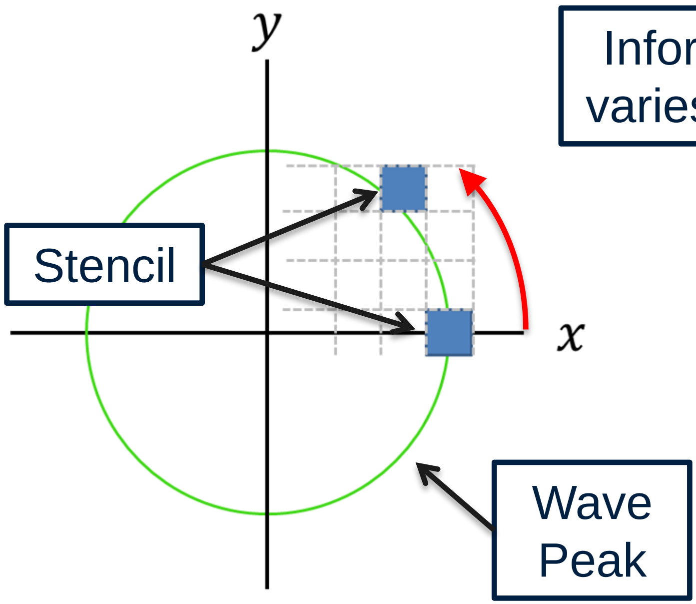
- Zalesak [4] proposed the first multidimensional flux limiter for FCT and also improved the clipping problem

Starting Point: Flux-corrected Transport (FCT)

Traditional flux limiters require a priori bounds to be place on the solution in each cell at each time step - usually calculated from spatial neighbors

Disadvantages:

1. No way to calculate “correct” upper and lower bounds ahead of time for multidimensional, nonscalar problems
2. Relying on information taken from spatial neighbors can introduce anisotropy



Information used to limit wave peak varies with position ⇒ ANISOTROPY

Can these disadvantages be avoided by a new approach?

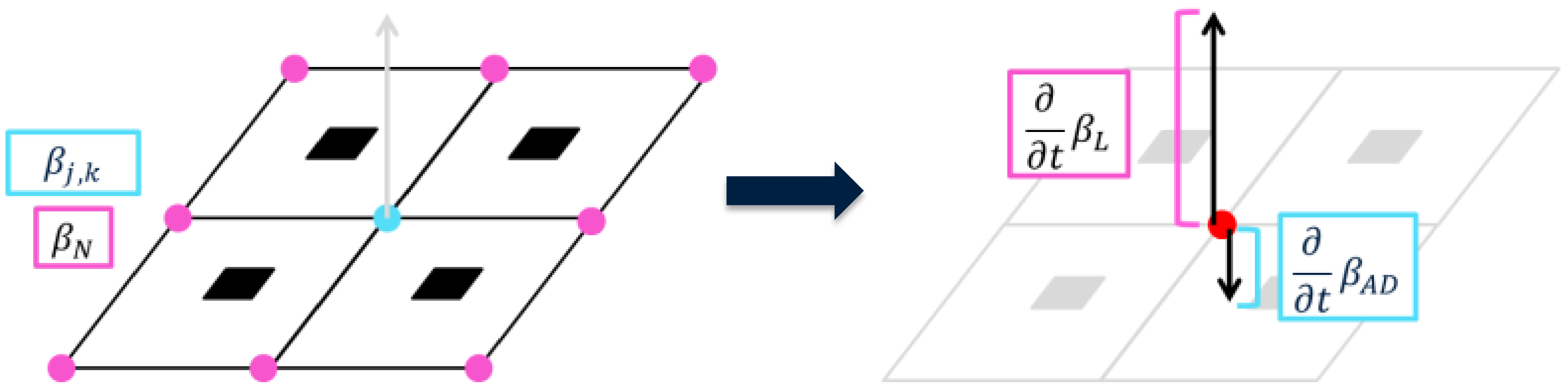
A Vorticity Preserving Approach to Limiting

Our new limiter is concerned primarily with temporal changes of the vertex fluxes and cannot introduce new anisotropy into the solution

- Nodal drivers, β, reflect the specific physics of the problem as expressed through the governing equations
- A first order driver and antidiffusion correction are first calculated by the isotropic base schemes

$$\frac{\partial}{\partial t}\beta_H = \frac{\partial}{\partial t}\beta_L + \frac{\partial}{\partial t}\beta_{AD}$$

$$\omega\left|\frac{\partial}{\partial t}\beta_{AD}\right| \leq f(\phi, |\nu|)\left|\frac{\partial}{\partial t}\beta_L\right|$$



A Vorticity Preserving Approach to Limiting

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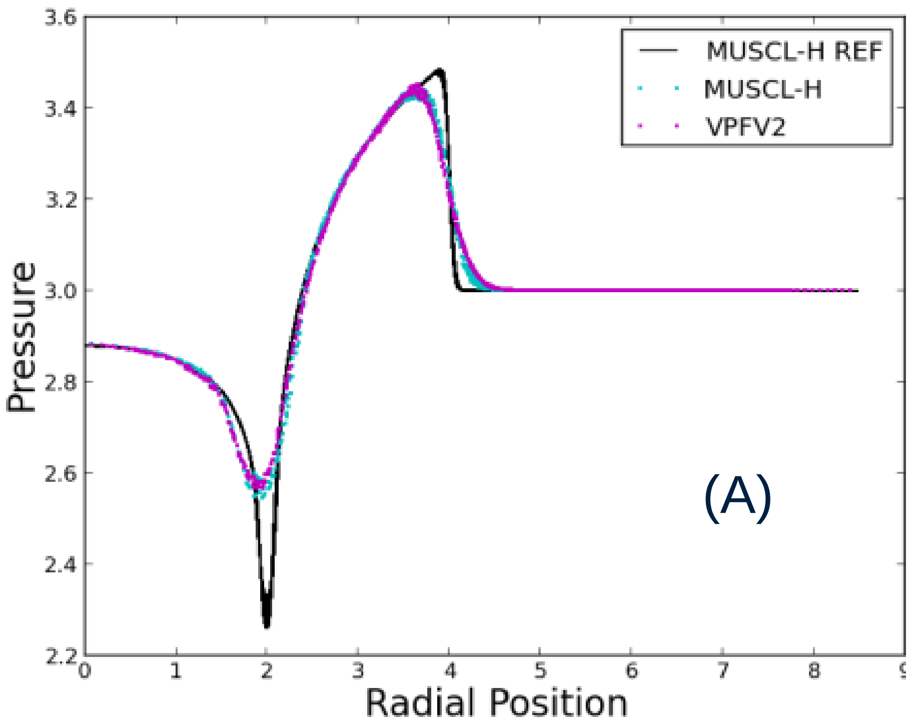
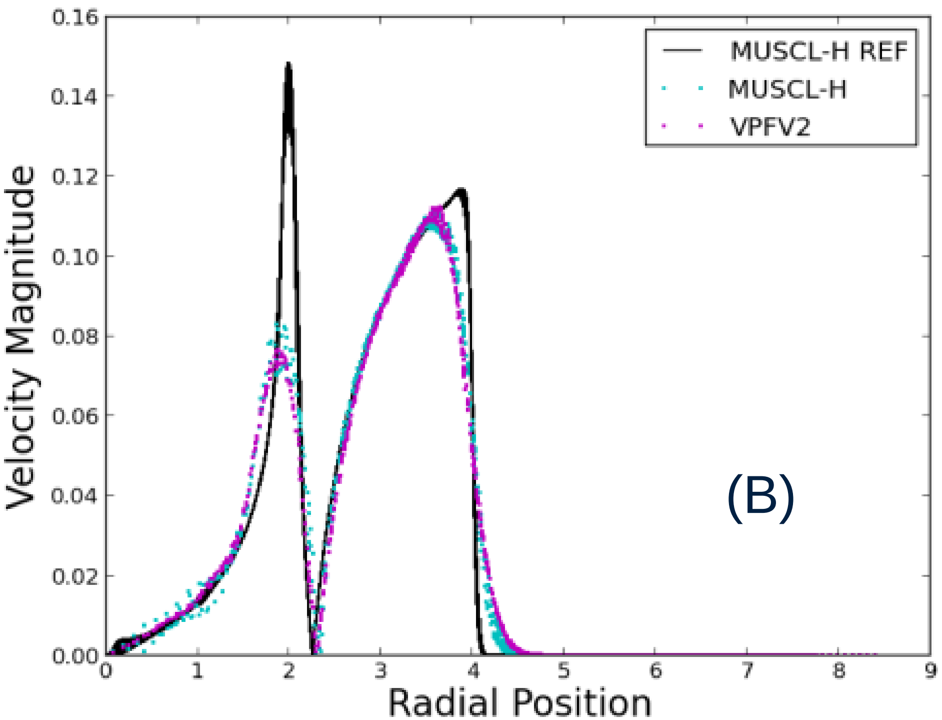
An "indicator quantity", ϕ, and the function $f(\phi, |\nu|)$ can affect the solution's phase and amplitude

- $f(\phi, |\nu|) \rightarrow f(|\nu|)$ forces the limiter to treat all data as the most difficult possible – even if not necessary
- experimental evidence shows that introducing ϕ as an empirical measure of complexity can alleviate excessive limiting
    - the difference is not very great, but seems to merit further investigation
    - current results will be presented

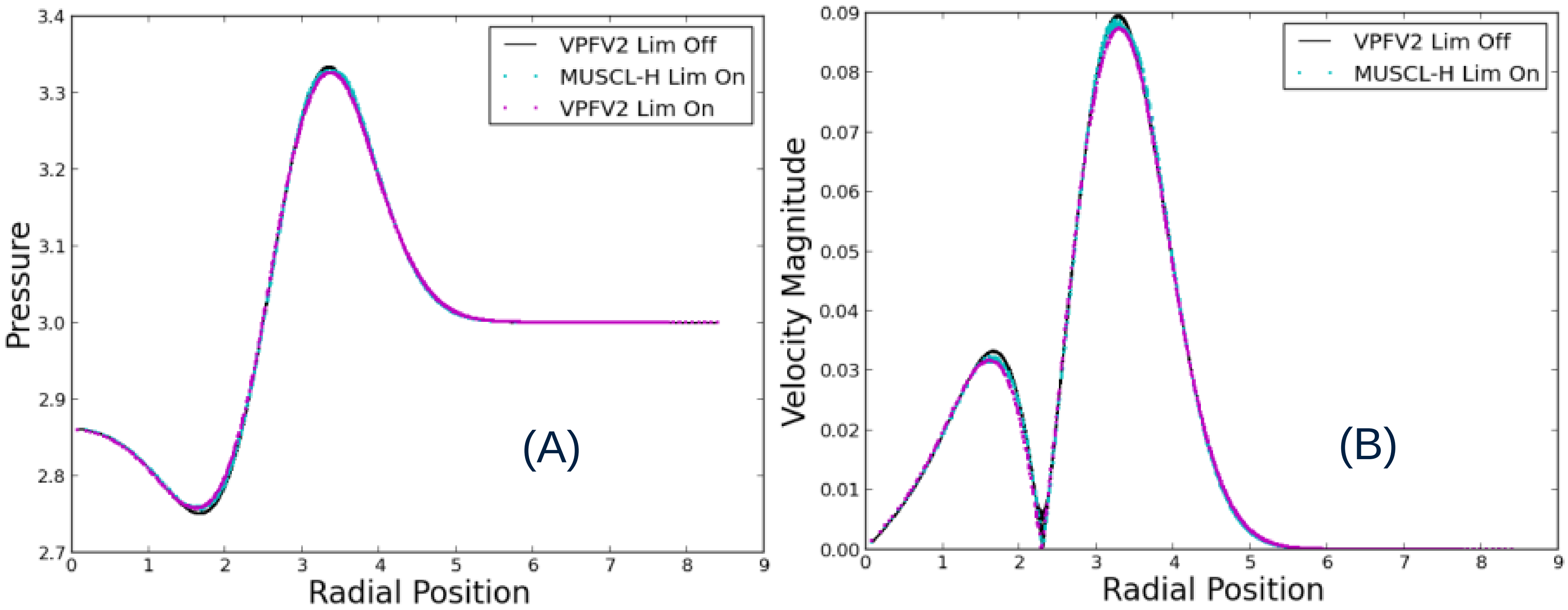
Limited Results





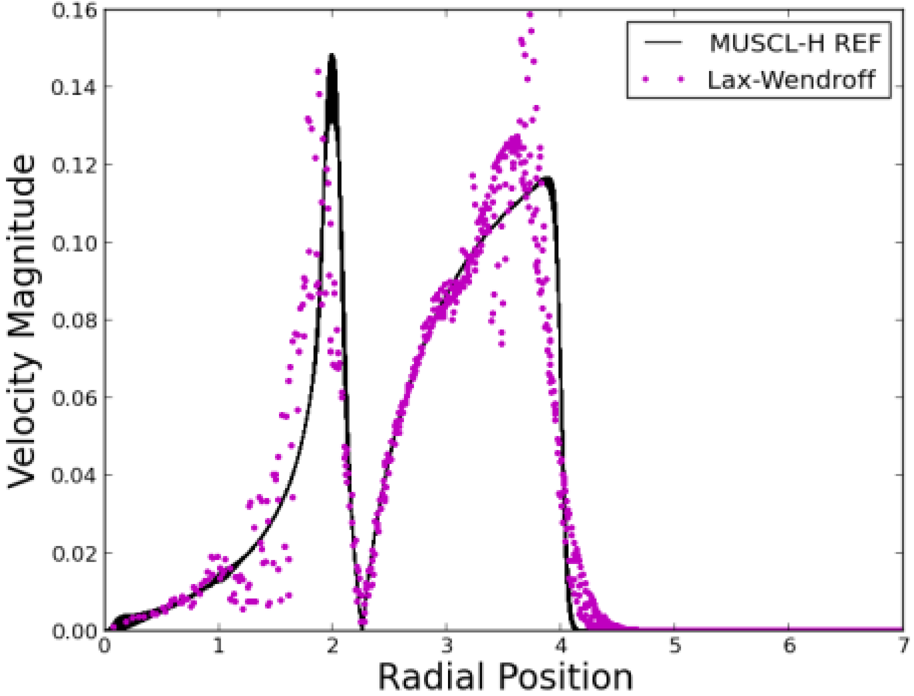
Comparison of MUSCL-H ($\nu = 0.4$) and VPFV2 with New Limiter ($\nu = 0.8$):
(A) Pressure (B) Velocity Magnitude

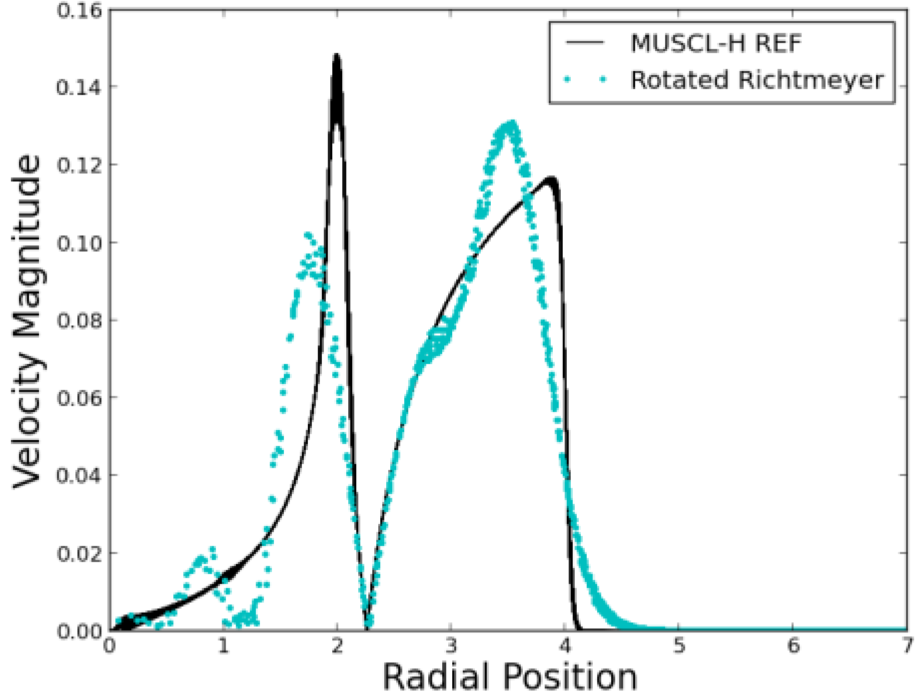
Limited Results - Smooth Problem



MUSCL-H ($\nu = 0.4$) and VPFV2 ($\nu = 0.8$), Gaussian Perturbation:
(A) Pressure (B) Velocity Magnitude

Progress Summary to Date





LW → RR

VORTICITY CONTROL

From Lax-Wendroff to VPFV2 with New Limiter

Progress Summary to Date

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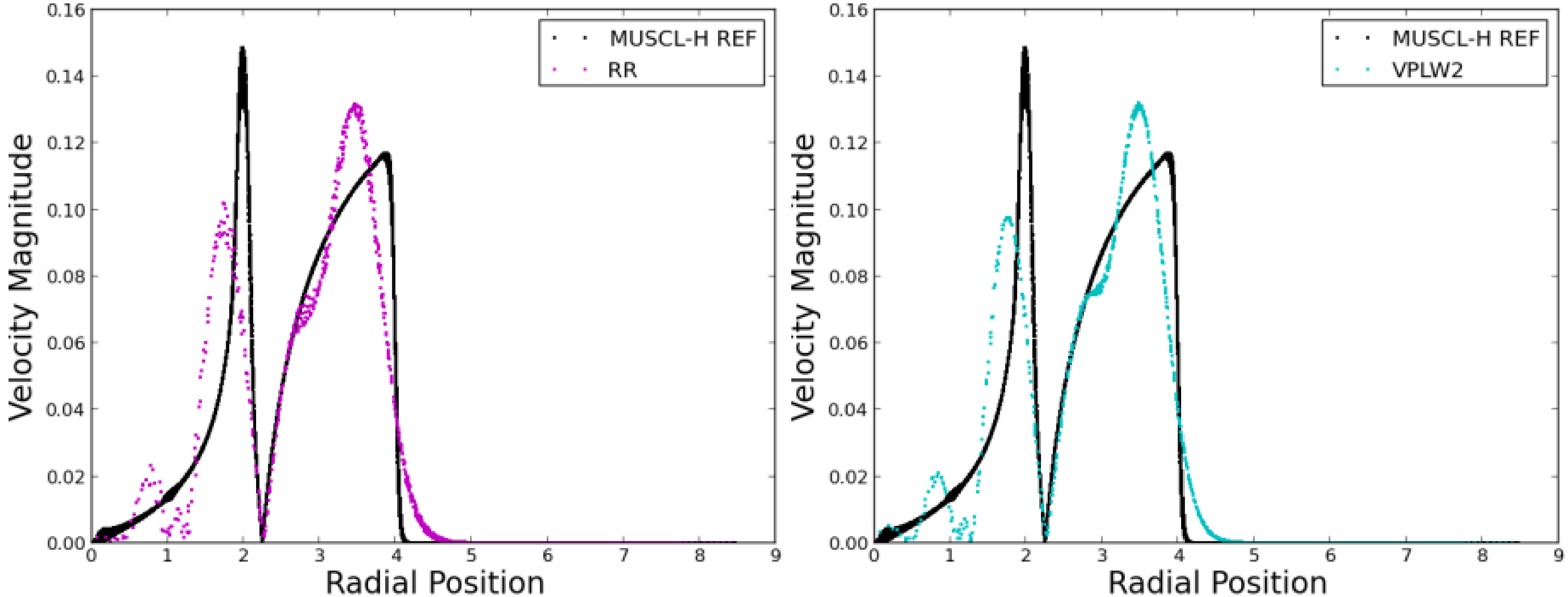

RR

VPLW2

DISPERSION ANALYSIS

From Lax-Wendroff to VPFV2 with New Limiter

Progress Summary to Date

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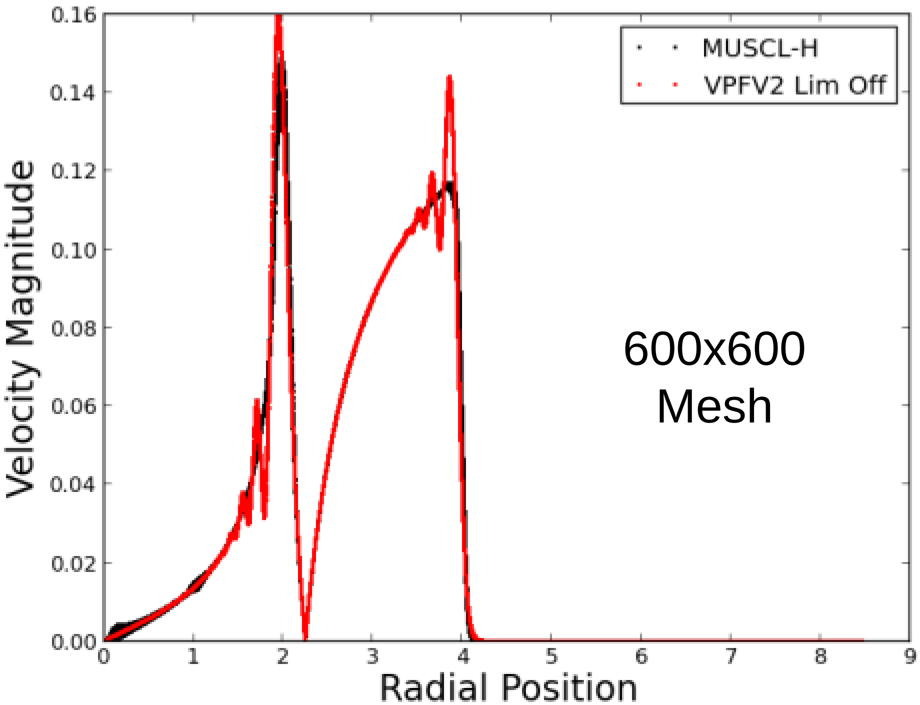

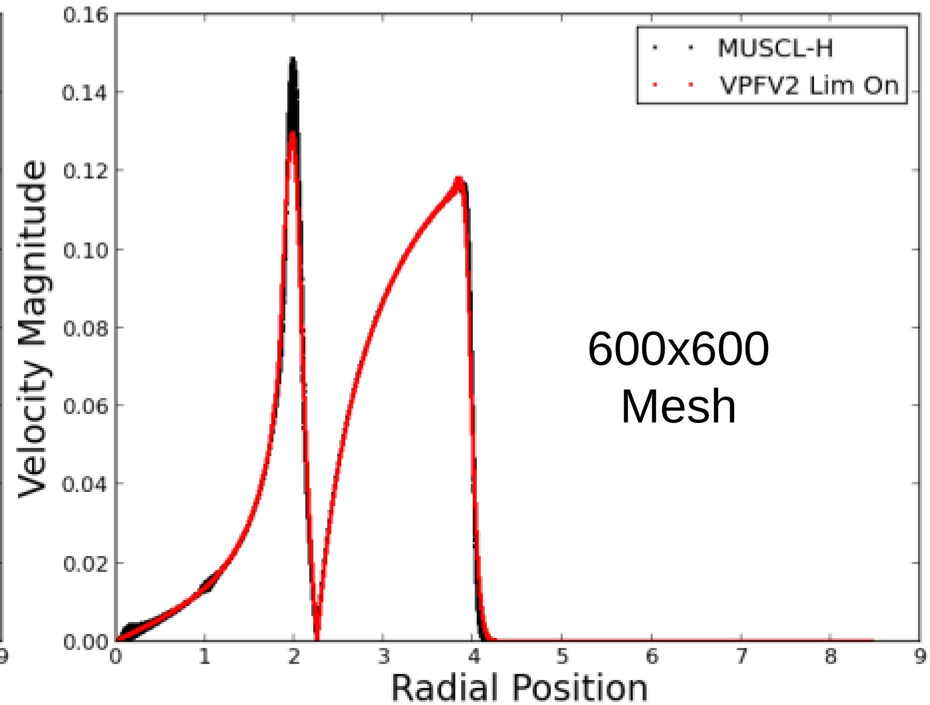

VPFV2 Limiter Off → VPFV2 Limiter On

NEW LIMITER

From Lax-Wendroff to VPFV2 with New Limiter

Conclusions

1. **Vertex fluxes enable vorticity to be preserved and isotropy to be improved**
2. **This requires a new form of limiting, which must be vertex based**
3. **This structure applies directly to Lagrangian grids**
4. **Within this framework, some flexibility remains that allows for detailed improvements**

Future Work and Acknowledgements

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1. Improvements to the limiting mechanism
2. Third order accuracy
3. Implement the method for the Euler equations and a Lagrangian grid

Works Cited

[1] Dukowicz, J. K. and Meltz, B. J. A., Journal of Computational Physics, Volume 99, Issue 1, March 1992, pp. 115-134

[2] Morton, K. W. and Roe, P.L., SIAM Journal on Scientific Computing, Vol. 23, No. 1, 2001, pp. 170-191

[3] Boris, J. P. and Book, D. L., Journal of Computational Physics, Volume 11, Issue 1, January 1973, pp. 38-69

[4] Zalesak, S. T., Journal of Computational Physics, Volume 31, Issue 3, June 1979, pp. 335-362

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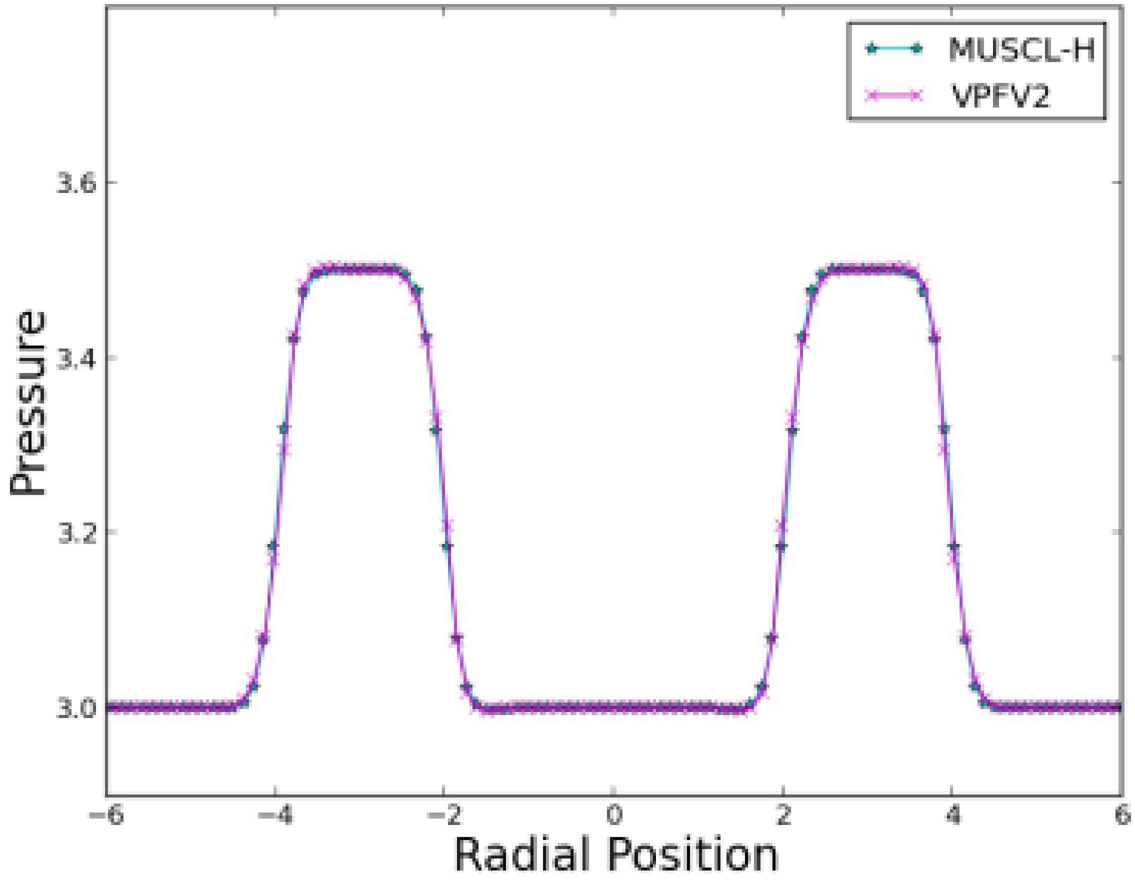
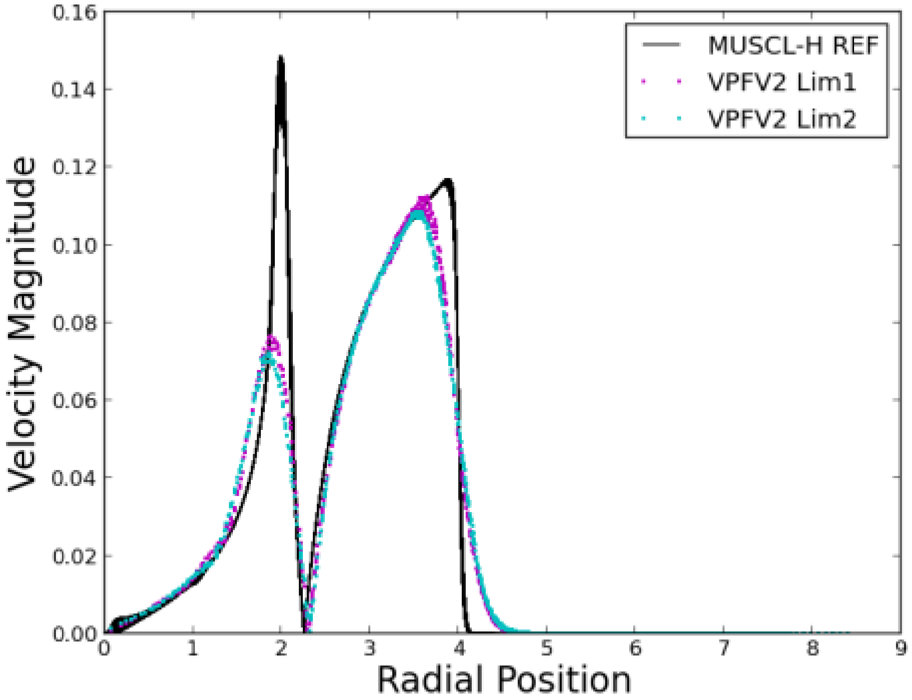
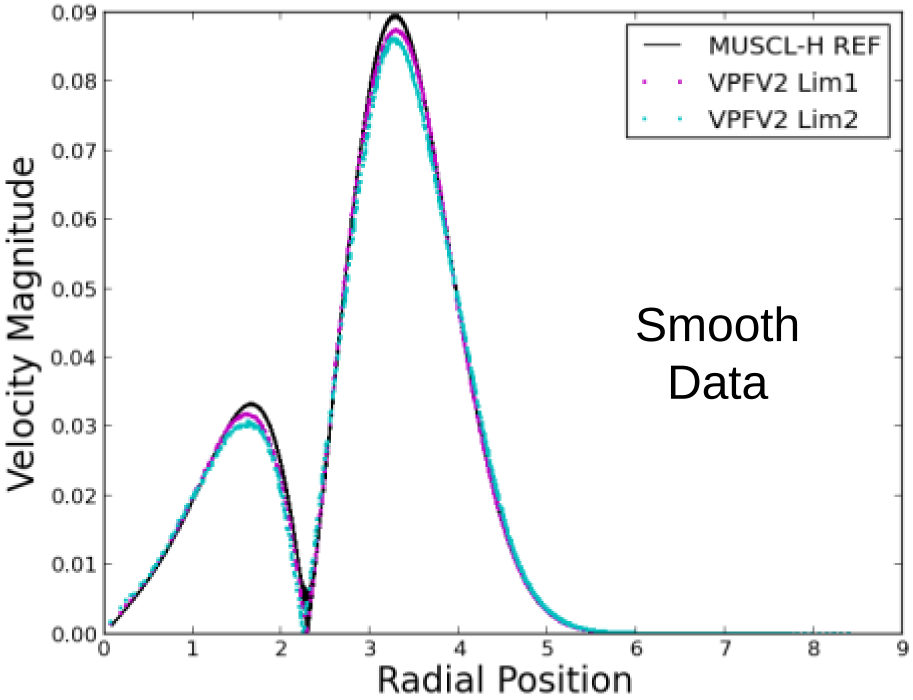
Limited Results - Square Wave



MUSCL-H ($\nu = 0.4$) and VPFV2 ($\nu = 0.8$): 1D Square Wave (initialized in 2D)

Limited Results - $f(|\nu|)$





VPFV2 $(\nu = 0.8)$