DE LA RECHERCHE À L'INDUSTRIE

An improved $k - \omega$ model for 3D hypersonic flows

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Plan

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1. Context & Equations
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2. Theoretical and numerical solvings
3. Results
   - Ramp at mach 5 (Delery 1990)
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4. Conclusions & Prospects

Context

Aim

- Our aim is to capture fluxes (pressure and thermal fluxes) in the flow over a blunt hypersonic body.
- Turbulence has a great impact on the values of these quantities so we need a robust and accurate model to do the job.
- Here we present a first-order turbulent model based on Wilcox $k - \omega$ model.
- We are only interested in solving stationnary states.

Incompressible $k - \omega$ turbulent equations

Equations for U (velocity), k (turbulent kinetic energy) and ω (characteristic frequency of the turbulence)

$$\partial_t U + U \cdot \nabla U + \nabla \left(P + \frac{2}{3} k \right) = \nabla \cdot \left[(\nu + \nu_t) \left(\nabla U + \nabla U^T \right) \right],$$

$$\nabla \cdot U = 0,$$

$$\partial_t k + U \cdot \nabla k = 2\nu_t (S : S) - \beta^* k\omega + \nabla \cdot [(\nu + \sigma^* \nu_t) \nabla k],$$

$$\partial_t \omega + U \cdot \nabla \omega = 2\alpha (S : S) - \beta\omega^2 + \nabla \cdot [(\nu + \sigma\nu_t) \nabla \omega],$$

$$\nu_t = \frac{k}{\omega}, \alpha = \frac{13}{25}, \beta = \beta_0 f_\beta \, , \, \beta^* = \beta_0^* f_\beta,$$

$$\sigma = \sigma^* = 0.5, \beta_0 = \frac{9}{125} \, , \, \beta_0^* = \frac{9}{100},$$

$$f_\beta = \frac{1 + 70\chi_\omega}{1 + 80\chi_\omega} \text{ où } \chi_\omega = \left| \frac{(\Omega \otimes \Omega) : S}{(\beta_0 \omega)^3} \right|,$$

$$f_\beta^* = \begin{cases} 1 & , \quad \chi_k \leq 0 \\ \dfrac{1 + 680\chi_k^2}{1 + 400\chi_k} & , \quad \chi_k \geq 0 \end{cases} , \quad \chi_k = \frac{1}{\omega^3} \nabla k \nabla \omega.$$

From incompressible flows towards compressible flows

Compressible equations

- Compressible equations are obtained by analogy with incompressible equations. Mean quantities are defined by pondering by the mass (Favre's averaging).
- We ensure in the code that Reynolds Tensor $R = (\nu + \nu_t)\left(\nabla U + \nabla U^T\right) - \frac{2}{3}k\mathbf{I}$ have all his diagonal coefficients negative and all non diagonal satisfy $\left|R_{ij}\right| \leq \sqrt{R_{ii}R_{jj}}$ because of its physical meaning ("realisibility condition").

Final system of equations (1)

$$\partial_t(\rho) + \nabla\cdot(\rho U) = 0,$$

$$\partial_t(\rho U) + \nabla(\rho U \otimes U) + \nabla\left(P + \frac{2}{3}\rho k\right)$$

$$= \nabla\cdot\left[\rho(\nu+\nu_t)\left(\nabla U + \nabla U^T - \frac{2}{3}(\nabla\cdot U)\mathbf{I}\right)\right],$$

$$\partial_t(\rho E) + \nabla\left(\left(\rho E + P + \frac{2}{3}\rho k\right)U\right) - \nabla\cdot\left(\left(\lambda + \frac{\rho\nu_t C_p}{Pr_t}\right)\nabla T\right)$$

$$= \nabla\cdot\left[\rho(\nu+\nu_t)\left(\nabla U + \nabla U^T - \frac{2}{3}(\nabla\cdot U)\mathbf{I}\right)U\right] + \nabla\cdot[\rho(\nu+\sigma^*\nu_t)\nabla k],$$

$$\partial_t(\rho k) + \nabla\cdot(\rho k U) = -f_c\rho R : \nabla U - \beta^*_{new}\rho k\omega + \nabla\cdot[\rho(\nu+\sigma^*\nu_t)\nabla k],$$

$$\partial_t(\rho\omega) + \nabla\cdot(\rho\omega U) = -\alpha f_c\rho R : \nabla U - \beta_{new}\rho\omega^2 + \nabla\cdot[\rho(\nu+\sigma\nu_t)\nabla\omega],$$

$$\nu_t = \alpha^*\frac{k}{\omega}, \nu = \frac{\mu}{\rho},$$

$$S = \frac{\nabla U + \nabla U^T}{2}, \Omega = \frac{\nabla U - \nabla U^T}{2}, R = \nu_t\left(2S - \frac{2}{3}(\nabla\cdot U)\mathbf{I}\right) - \frac{2}{3}k\mathbf{I},$$

Final system of equations (2)

$$f_c = 1.7, Re_t = \frac{k}{\nu\omega}, \sigma = \sigma^* = 0.5, \beta_0 = \frac{9}{125}, \beta_0^* = \frac{9}{100}, \alpha_0^* = \frac{1}{3}\beta_0, \alpha_0 = \frac{1}{9},$$

$$\alpha^* = \frac{\alpha_0^* + Re_t/R_k}{1 + Re_t/R_k}, \alpha = \frac{13}{25}\frac{\alpha_0 + (Re_t/R_\omega)}{1 + (Re_t/R_\omega)}(\alpha^*)^{-1},$$

$$\beta = \beta_0 \frac{4/15 + (Re_t/R_\beta)^4}{1 + (Re_t/R_\beta)^4} f_\beta, \beta^* = \beta_0^* f_\beta,$$

$$\beta^*{}_{new} = \beta^* - 1.5 \max(Ma_t^2 - Ma_{t_0}^2, 0)\beta \qquad \text{où} \qquad Ma_{t_0} = 0.5$$

$$\beta_{new} = \beta(1 + 1.5 \max(Ma_t^2 - Ma_{t_0}^2, 0))$$

$$R_\beta = 8\,,\ R_k = 6\,,\ R_\omega = 6$$

$$f_\beta = \frac{1 + 70\chi_\omega}{1 + 80\chi_\omega} \qquad \text{où} \qquad \chi_\omega = \left| \frac{(\Omega \otimes \Omega) : S}{(\beta_0 \omega)^3} \right|,$$

$$f_\beta^* = \begin{cases} 1 & , \quad \chi_k \leq 0 \\ \dfrac{1 + 680\chi_k^2}{1 + 400\chi_k} & , \quad \chi_k \geq 0 \end{cases} \quad , \quad \chi_k = \frac{1}{\omega^3}\nabla k \nabla \omega,$$

$$Pr_t = 0.7.$$

Boundary conditions

Boundary layer equations

$$\partial_t(\rho k) + \nabla \cdot (\rho k U) = \rho R : \nabla U - \beta^* \rho k \omega + \nabla \cdot \left[\rho\left(\nu + \sigma^* \nu_t\right) \nabla k\right],$$
$$\partial_t(\rho \omega) + \nabla \cdot (\rho \omega U) = \alpha \rho R : \nabla U - \beta \rho \omega^2 + \nabla \cdot \left[\rho\left(\nu + \sigma \nu_t\right) \nabla \omega\right].$$

If one looks for $k \sim C_k y^n$ and $\omega \sim \dfrac{C_\omega}{y^n}$ so that dissipation $\epsilon = k\omega$ tends to a finite value when the wall distance $y$tends to zero, one gets:

$$0 = -\beta^* C_\omega C_k + \nu(n(n-1)) C_k y^{n-2}, \quad (1)$$
$$0 = -\beta C_\omega^2 y^{-2n} + \nu(n(n+1)) C_\omega y^{-n-2}. \quad (2)$$

Equation (1) gives $n = 2$ and $\beta^* C_\omega = 2\nu$.
Zquation (2) is satisfied if and only if $-2n = n - 2$ which also means $n = 2$ and $\beta C_\omega = 6\nu$.

Plan

Some mathematical properties of a simplified incompressible model(1)

$$\partial_t U + U \cdot \nabla U + \nabla \left(P + \frac{2}{3} k \right) = \nabla \cdot \left[(\nu + \nu_t) \left(\nabla U + \nabla U^T \right) \right],$$
$$\nabla \cdot U = 0,$$
$$\partial_t k + U \cdot \nabla k = 2\nu_t (S : S) - \beta^* k\omega + \nabla \cdot [(\nu + \sigma^* \nu_t) \nabla k],$$
$$\partial_t \omega + U \cdot \nabla \omega = 2\alpha (S : S) - \beta \omega^2 + \nabla \cdot [(\nu + \sigma \nu_t) \nabla \omega],$$
$$\nu_t = \frac{k}{\omega}, \alpha = \frac{13}{25}, \sigma = \sigma^* = 0.5, \beta = \frac{9}{125}, \beta^* = \frac{9}{100}.$$

If one looks for regular local in time 3D solutions to the $k - \omega$ model (in Sobolev spaces) on bounded regular domains we have the following properties:

Framework of the work

- we suppose that U is in the Sobolev space H_0^s for small times with $s > 4 + 3/2$.
- we suppose that k is in the Sobolev space H_0^s for small times with $s > 4 + 3/2$. item we suppose that $\omega - \omega_0$ is in the Sobolev space H_0^s for small times with $s > 4 + 3/2$ where ω_0 is a constant (the boundary condition on the domain).

Some properties of a simplified incompressible model(2)

Then we have

A priori estimates

There exists some integer n depending on s such that

$$\frac{d}{dt}\left(||U||^2{}_{H^s} + ||k||^2{}_{H^s} + ||\omega - \omega_0||^2{}_{H^s}\right) \leq \left(||U||^2{}_{H^s} + ||k||^2{}_{H^s} + ||\omega - \omega_0||^2{}_{H^s}\right)^{n(s)}$$

so that if one finds ways to construct solutions through an iterative process there will exist local in time smooth solutions in 3D. (Mathiaud 2008 :“Local smooth solutions of the incompressible $k - \epsilon$ model and the low turbulent diffusion limit)

System of turbulent equations

The global system of compressible equations can be under the following form:

$$\partial_t V + \partial_x F(V) + \partial^2_{xx} G(V) = S(V) \tag{3}$$

where V is the vector $\begin{pmatrix} \rho \\ \rho U \\ \rho E \\ \rho k \\ \rho\omega \end{pmatrix}$, $F(V)$ represents the flux associated to V, $G(V)$ the diffusive part of the system and $S(V)$ the source term.

Numerical solving of the stationnary states

The time marching is solved through:

$$\frac{V^{n+1} - V^n}{dt} + \partial_x F(V^{n+1}) + \partial^2_{xx} G(V^{n+1}) = S(V^{n+1}) \tag{4}$$

We use a finite volume scheme for Navier-Stokes equations with a Roe-type solver.

We note

$$\Delta(\rho k) = \rho^{n+1} k^{n+1} - \rho^n k^n, \tag{5}$$
$$\Delta(\rho\omega) = \rho^{n+1} \omega^{n+1} - \rho^n \omega^n, \tag{6}$$
$$\Delta\rho = \rho^{n+1} - \rho^n, \tag{7}$$
$$\Delta(\rho U) = \rho^{n+1} U^{n+1} - \rho^n U^n. \tag{8}$$

Implicitation of k and ω equations

If one linearizes turbulent kinetic equation around the state at time n one gets:

$$\begin{aligned}
& \frac{\Delta \rho k}{dt} + \nabla \cdot (U^n \rho^n k^n) + \nabla \cdot (U^n \Delta \rho k) + \nabla \cdot (k^n \Delta \rho U) + \nabla \cdot \left(-\frac{U^n k^n}{(\rho^n)^2} \Delta \rho \right) \\
= \; & (-f_c \rho R : \nabla U)^n - (\beta^*_{new})^n \rho^n k^n \omega^n - (\beta^*_{new})^n \omega^n \Delta \rho k - (\beta^*_{new})^n k^n \Delta \rho \omega \\
+ \; & \nabla \cdot \left[(\rho (\nu + \sigma^* \nu_t))^n \nabla k^n \right] + \nabla \cdot \left[(\nu + \sigma^* \nu_t)^n \nabla (\Delta \rho k) \right] \\
- \; & \nabla \cdot \left[\left((\nu + \sigma^* \nu_t)^n k^n \right) \nabla (\Delta \rho) \right]
\end{aligned}$$

Then turning to ω equation:

$$\begin{aligned}
& \frac{\Delta \rho \omega}{dt} + \nabla \cdot (U^n \rho^n \omega^n) + \nabla \cdot (U^n \Delta \rho \omega) + \nabla \cdot (\omega^n \Delta \rho U) + \nabla \cdot \left(-\frac{U^n \omega^n}{(\rho^n)^2} \Delta \rho \right) \\
= \; & (-\alpha f_c \rho R : \nabla U)^n - (\beta^*_{new})^n \rho^n \omega^n \omega^n - (\beta^*_{new})^n \omega^n \Delta \omega - (\beta^*_{new})^n \omega^n \Delta \omega \\
+ \; & \nabla \cdot \left[(\rho (\nu + \sigma^* \nu_t))^n \nabla \omega^n \right] + \nabla \cdot \left[(\nu + \sigma^* \nu_t)^n \nabla (\Delta \rho \omega) \right] \\
- \; & \nabla \cdot \left[\left((\nu + \sigma^* \nu_t)^n \omega^n \right) \nabla (\Delta \rho) \right]
\end{aligned}$$

Stationnary state

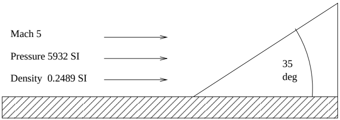
Looking for stationnary states

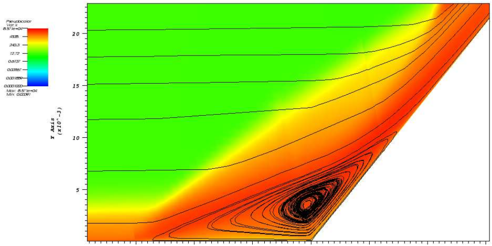
$$\frac{\Delta\rho k}{dt} + \nabla\cdot\left(U^n\rho^n k^n\right) + \nabla\cdot\left(U^n\Delta\rho k\right) + \nabla\cdot\left(k^n\Delta\rho U\right) + \nabla\cdot\left(-\frac{U^n k^n}{(\rho^n)^2}\Delta\rho\right)$$
$$= \left(-f_c\rho R:\nabla U\right)^n - (\beta^*_{new})^n\rho^n k^n\omega^n - (\beta^*_{new})^n\omega^n\Delta\rho k - (\beta^*_{new})^n k^n\Delta\rho\omega$$
$$+ \nabla\cdot\left[\left(\rho\left(\nu+\sigma^*\nu_t\right)\right)^n\nabla k^n\right] + \nabla\cdot\left[\left(\nu+\sigma^*\nu_t\right)^n\nabla(\Delta\rho k)\right]$$

$$\frac{\Delta\rho\omega}{dt} + \nabla\cdot\left(U^n\rho^n\omega^n\right) + \nabla\cdot\left(U^n\Delta\rho\omega\right) + \nabla\cdot\left(\omega^n\Delta\rho U\right) + \nabla\cdot\left(-\frac{U^n\omega^n}{(\rho^n)^2}\Delta\rho\right)$$
$$= \left(-\alpha f_c\rho R:\nabla U\right)^n - (\beta^*_{new})^n\rho^n\omega^n\omega^n - (\beta^*_{new})^n\omega^n\Delta\omega - (\beta^*_{new})^n\omega^n\Delta\omega$$
$$+ \nabla\cdot\left[\left(\rho\left(\nu+\sigma^*\nu_t\right)\right)^n\nabla\omega^n\right] + \nabla\cdot\left[\left(\nu+\sigma^*\nu_t\right)^n\nabla(\Delta\rho\omega)\right]$$

Plan

2D ramp at mach 5 (Delery 1990)





2D ramp

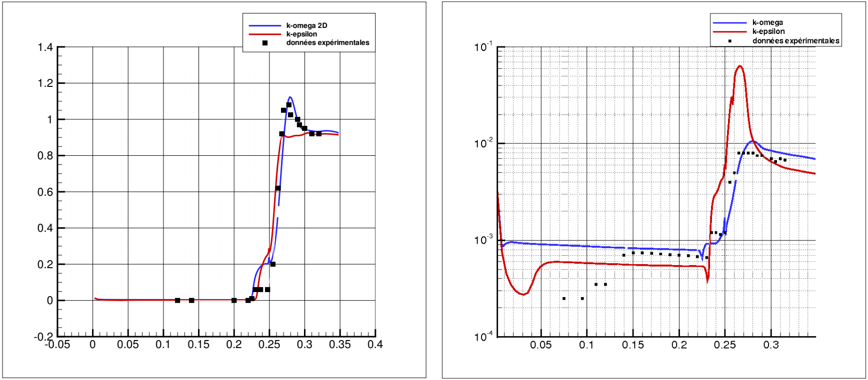
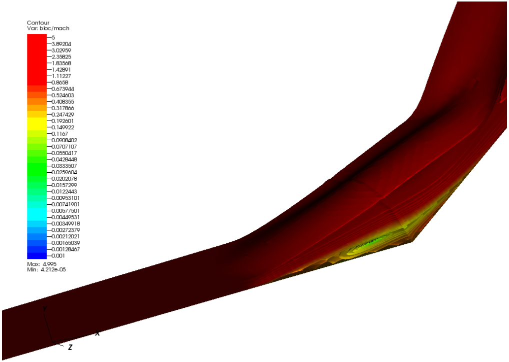


Figure: Pressure coefficient & Stanton Number

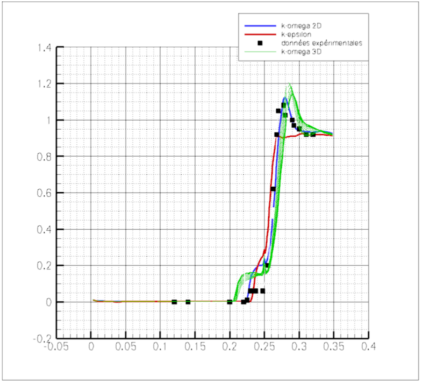
$$C_p = \frac{P - P_\infty}{\frac{1}{2}\rho_\infty V_\infty^2}, \qquad St = \frac{\lambda \nabla T \cdot \vec{n}}{\rho_\infty V_\infty C \left(T_\infty (1 + \frac{\gamma - 1}{2} M_\infty^2) - T_p \right)}$$

3D results



Contour
Var: bloc/mach
5
3.89204
3.02959
2.35825
1.83568
1.42891
1.11227
0.8658
0.673944
0.524603
0.408355
0.317866
0.247429
0.192601
0.149922
0.1167
0.0908402
0.0707107
0.0550417
0.0428448
0.0333507
0.0259604
0.0202078
0.0157299
0.0122443
0.00953101
0.00741901
0.00577501
0.00449531
0.00349918
0.00272379
0.00212021
0.00165039
0.00128467
0.001
Max: 4.995
Min: 4.212e-05
X
Z

2D .vs. 3D



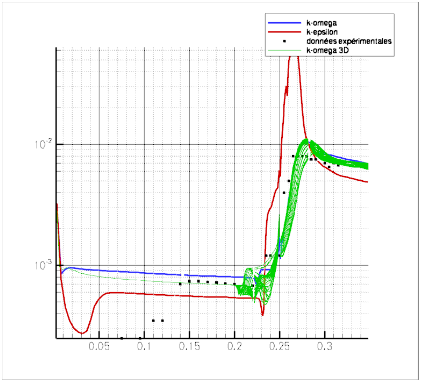
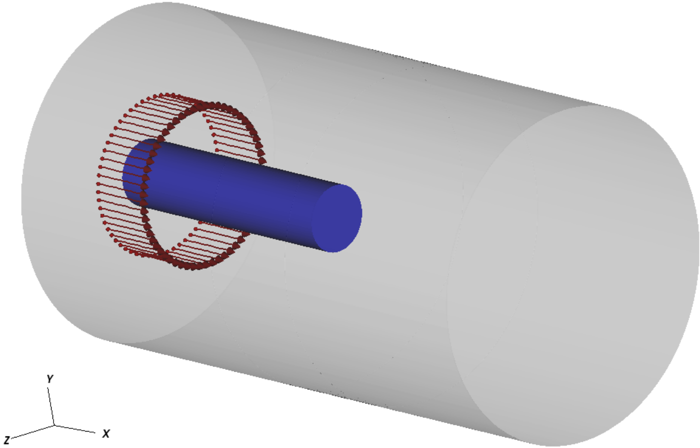


Figure: Pressure coefficient & Stanton Number

3D results are quite in agreement with 2D results

Dutton and Herrin experiment(1994, AIAA Journal Vol. 32, No. 1, January)



Mach = 2.46, Pressure = 32078.5, Density =0.84302, Velocity= 567.8

Mach Number and Turbulent intensity

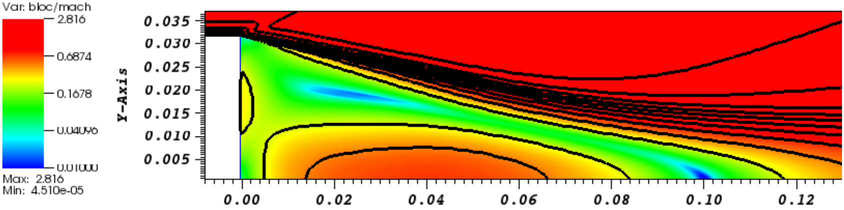


Figure: Recirculation point measured at 0.084m by Dutton and assessed at 0.1m by the code

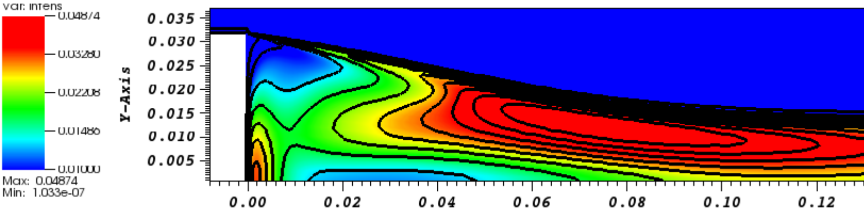


Figure: Ratio between k and $\frac{1}{2}u^2$: the max was measured at 0.044 by Dutton

Pressure results for different models

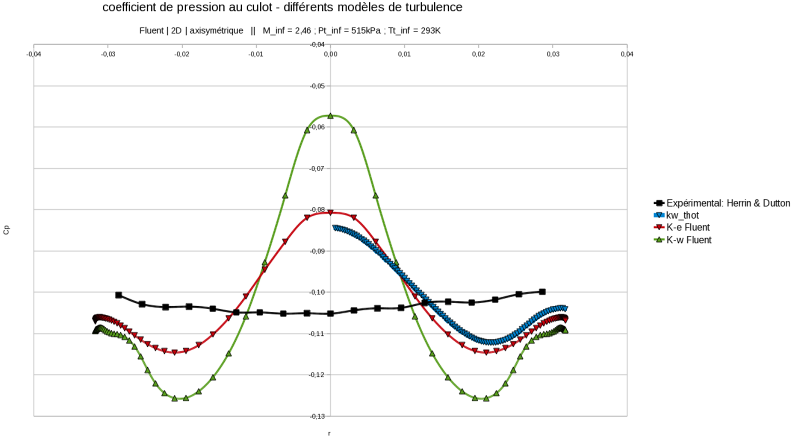


Figure: C_p on the wall

2D (5.10^4 cells) .vs. 3D (10^7 cells)

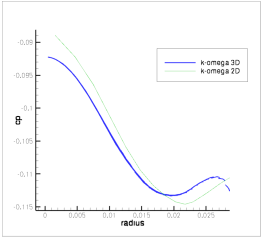


Figure: C_p on the wall

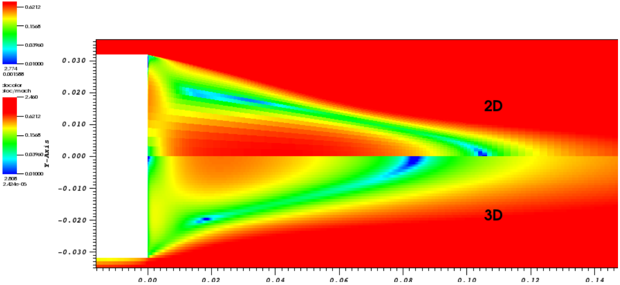


Figure: Reattachment point

Why such a difference?(1)

Putting rough boundary conditions

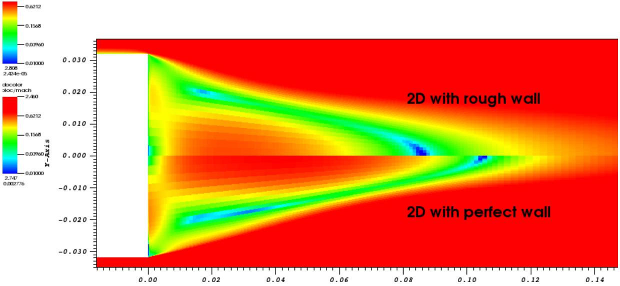


Figure: Reattachment point

Why such a difference?(2)

Flaws in the mesh creating orthoradial velocity

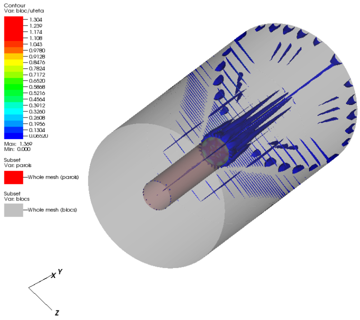


Figure: Flaws in the mesh creating orthoradial velocity

Plan

Conclusions & Prospects

- The $k - \omega$ has been improved to provide good agreements with experiments.
- Theoretical and Numerical analysis have been done.
- Understanding 3D results are still under investigation. Fortunately we are able to perform computations very rapidly (3 hours on 1000 processors for 10^7 cells.

2D Dutton experiment: data

```
|==========================================|
|         Aerodynamic conditions           |
|==========================================|
 Initial wind tunnel conditions:
 ------------------------------
 Mach number            Mach  =        2.46
 Stag. pressure         pi    =   515000.00 Pa
 Stag. temperature      Ti    =      293.00 K

 Reference lenght and area:
 --------------------------
 Reference lenght       Lref  =     0.25400 m
 Reference surface      Sref  =     0.00317 m2

 Gas model (perfect gas):
 ------------------------
 Spec. heat ratio       Gamma =       1.400
 Ideal gas constant     rgas  =     287.053 J/kg.K
 Specific heat (cst V)  Cv    =     717.633 J/kg.K
 Specific heat (cst p)  Cp    =    1004.686 J/kg.K

 Ideal gas cst / mole   Rgas  =     8.31432 J/(mole.K)
 Molecular weight       Mair  =     0.02896 kg/mole

 Wind tunnel conditions:
 -----------------------
 Temperature            T     =      132.56 K
 Pressure               p     =    32078.50 Pa
 Density                ro    =     0.84302 kg/m**3

 Velocity and sound speed:
 -------------------------
 Velocity magnitude     V     =      567.79 m/s
                        V     =     2044.04 km/h
 Sound speed            a     =      230.81 m/s
                        a     =      830.91 km/h

 Viscosity model (sutherland law):
 ---------------------------------
 Suth. constant         Ts    =      110.40 K
 Ref. temperature       Tref  =      273.16 K
 Ref. viscosity         Muref =   0.172E-04 kg/(m.s)

 Molecular viscosity    Mu    =   0.916E-05 kg/(m.s)
 Kinematic viscosity    Nu    =   0.109E-04 m2/s

 Reynolds number:
 ----------------
 Reynolds number / m    Re/m  =   52.27 x 10+6
 Reynolds number / Lref Re/L  =   13.28 x 10+6
```


```
 Velocity and sound speed:
 -------------------------
 Velocity magnitude     V     =      567.79 m/s
                        V     =     2044.04 km/h
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 Reynolds number:
 ----------------
 Reynolds number / m    Re/m  =   52.27 x 10+6
 Reynolds number / Lref Re/L  =   13.28 x 10+6
```


data from Dutton(1991)

2D Dutton experiment: mesh

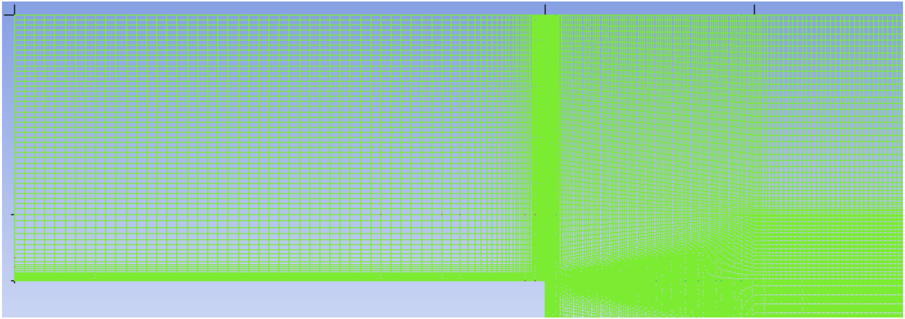
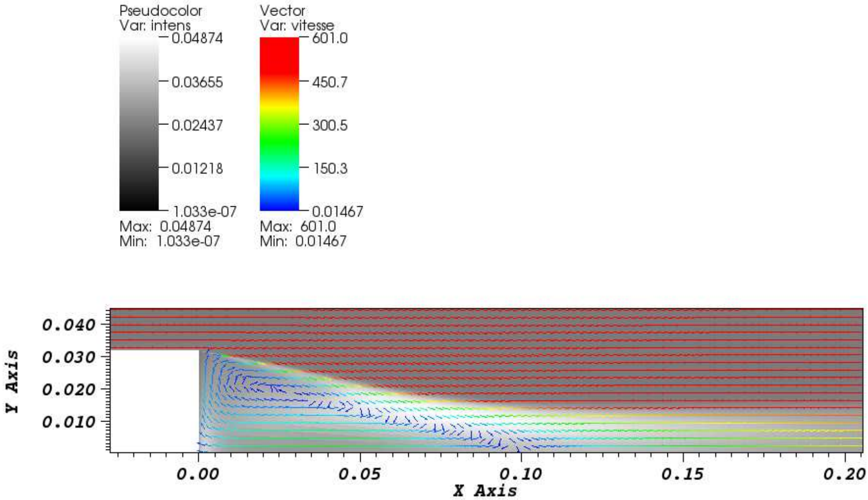


Figure: Icem mesh with 47888 nodes (R=32mm)

Velocity field



Reattachment zone for 2D models

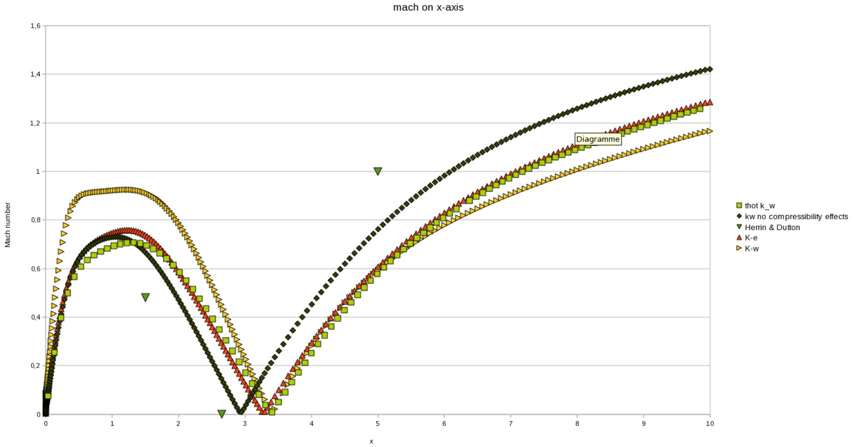


Figure: Mach number: CEA $k-w$,Fluent $k-\omega$ model, Herrin & Dutton experimental results, $k-\epsilon$ model in Fluent, $k-\omega$ model in Fluent)