Artificial Viscosity: Back to Basics

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Outline

- Historical review and reprise
- The standard view of viscosity
- Reconsideration
- Important Details
- Results

The muddy history of artificial viscosity

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- **Artificial viscosity is commonly attributed to Von Neumann and Richtmyer's 1950 paper**
  - *Too often this is truncated to Von Neumann alone*
  - *That paper had three main ideas: a finite difference solution to the PDEs, artificial viscosity & a stability analysis.*
  - *Two of these are Von Neumann's brainchildren, one is Richtmyer's.*
- **The original artificial viscosity was developed by Richtmyer (alone) in 1948 at Los Alamos**
- **The 1950 version is given as a pseudo-pressure, while the 1948 version is a viscous force.**
- **The forms are subtly different, do the details matter?**

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The first "hydro" calculations

- **The first hydrodynamic calculation was described in a Los Alamos report (LA-94) on June 20, 1944 – lead author Hans Bethe**
  - Feynmann was the calculational lead and marked the transition from human computers to IBM machines (done in April/May '44).
  - They used two methods to compute shocks, but only one of them worked well (the shock fitting by Peierls).
  - The finite difference method produced severe post-shock "wiggles" explained as thermal excitation.

- **Thereafter calculations were 1-D and Lagrangian, shocks were tracked until 1948.**
- **Von Neumann developed a "simple" finite difference method at Aberdeen (published a report on March 20, 1944).**

The artificial viscosity paper by Von Neumann and Richtmyer, J. Appl. Phys. 1950

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A Method for the Numerical Calculation of Hydrodynamic Shocks

J. VonNeumann and R. D. Richtmyer
Institute for Advanced Study, Princeton, New Jersey
(Received September 26, 1949)

The equations of hydrodynamics are modified by the inclusion of additional terms which greatly simplify the procedures needed for stepwise numerical solution of the equations in problems involving shocks. The quantitative influence of these terms can be made as small as one wishes by choice of a sufficiently fine mesh for the numerical integrations. A set of difference equations suitable for the numerical work is given, and the condition that must be satisfied to insure their stabilty is derived.

I. INTRODUCTION

In the investigation of phenomena arising in the flow of a compressible fluid, it is frequently desirable to solve the equations of fluid motion by stepwise numerical procedures, but the work is usually severely complicated by the presence of shocks. The shocks manifest themselves mathematically as surfaces on which density, fluid velocity, temperature, entropy and the like have discontinuities; and clearly the partial differential equations governing the motion require boundary conditions connecting the values of these quantities on the two sides of each such surface. The necessary boundary

(but preferably somewhat larger than) the spacing of the points of the network. Then the differential equations (more accurately, the corresponding difference equations) may be used for the entire calculation, just as though there were no shocks at all. In the numerical results obtained, the shocks are immediately evident as near-discontinuities that move through the fluid with very nearly the correct speed and across which pressure, temperature, etc. have very nearly the correct jumps.

It will be seen that for the assumed form of dissipation (and, indeed, for many others as well), the Rankine-Hugoniot equations are satisfied, provided the thick-

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LA-671, The first description of artificial viscosity written by Richtmyer (only!)

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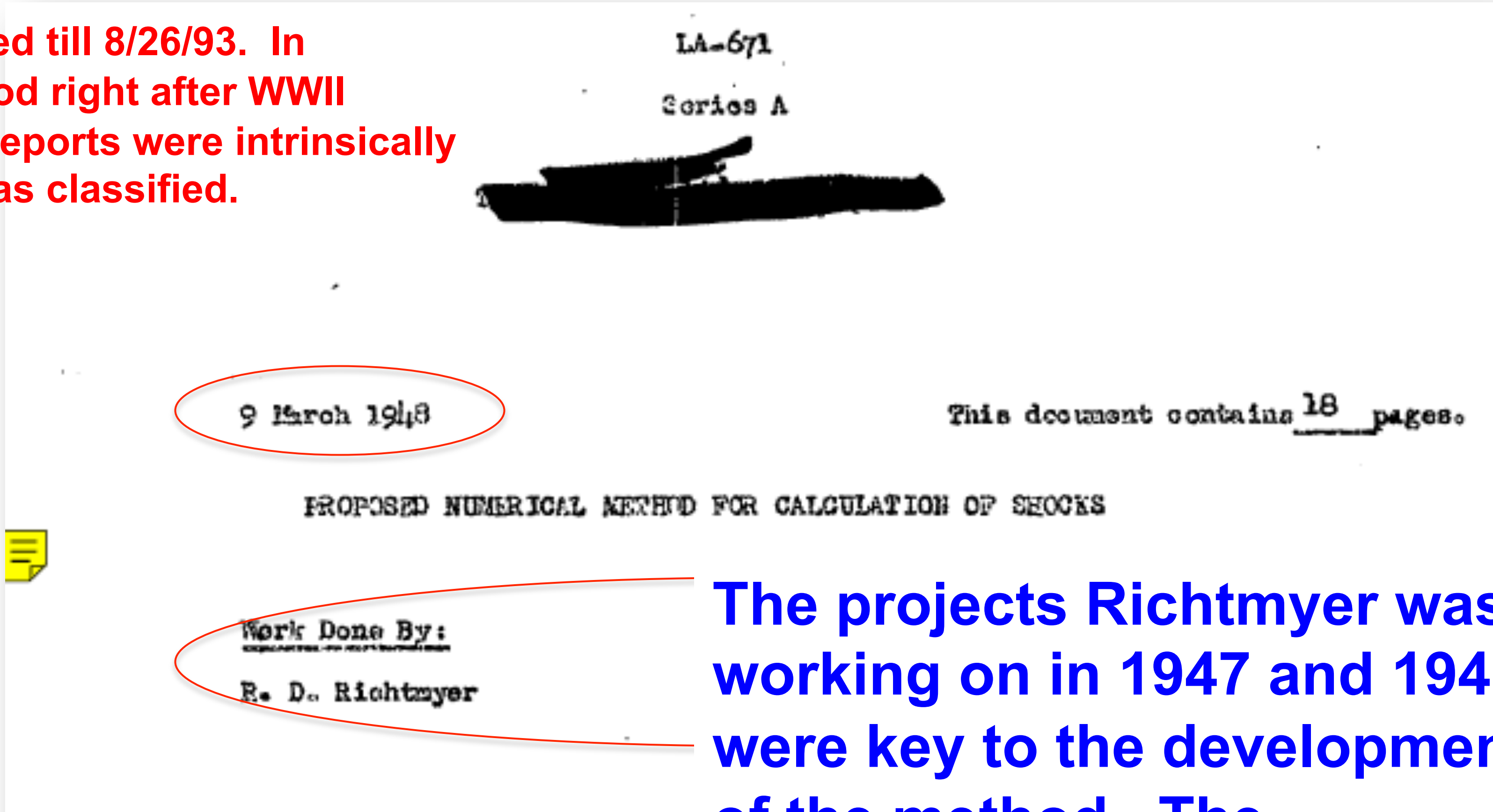
Classified till 8/26/93. In the period right after WWII all Lab reports were intrinsically treated as classified.


LA-671

Series A

[redacted]

9 March 1948

This document contains 18 pages.

PROPOSED NUMERICAL METHOD FOR CALCULATION OF SHOCKS

Work Done By:

R. D. Richtmyer

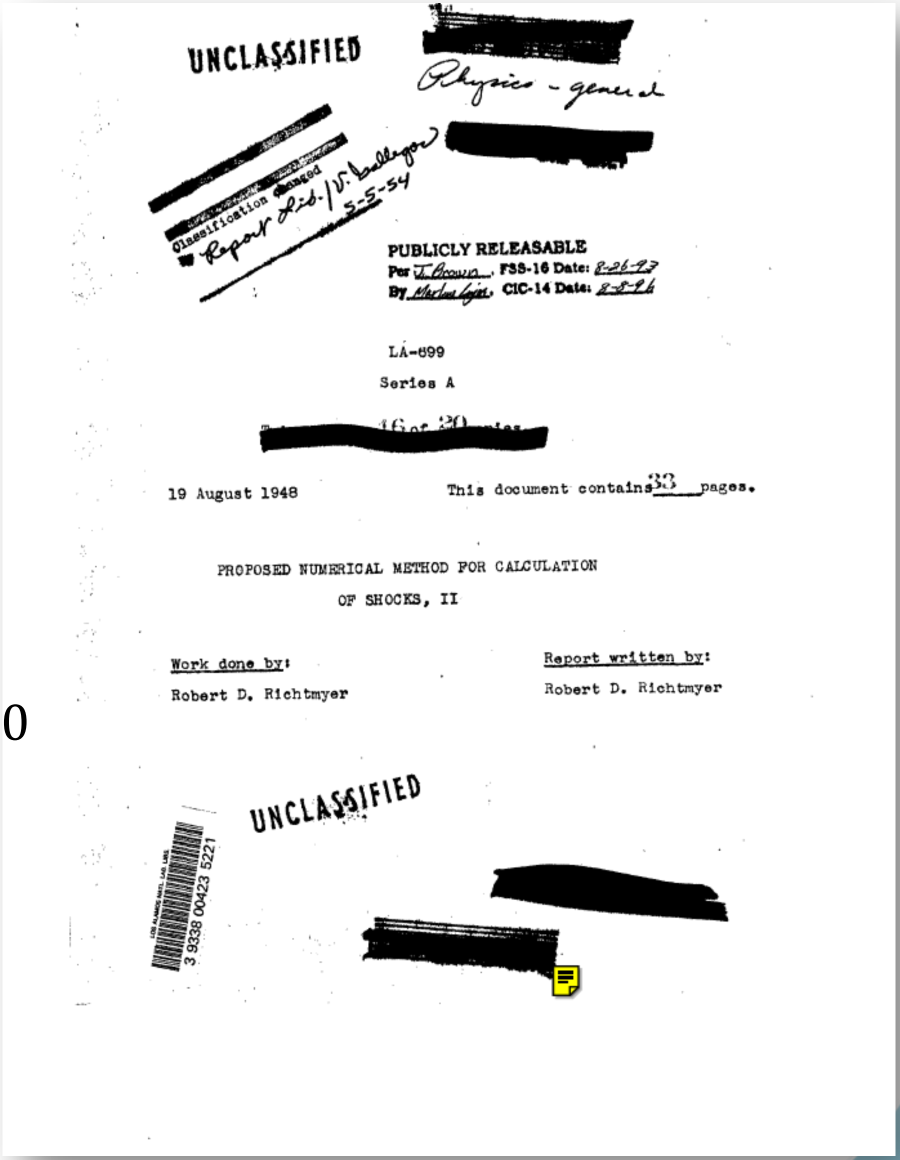
The projects Richtmyer was working on in 1947 and 1948 were key to the development of the method. The application was too complex for shock fitting.

Richtmyer published a second report five months later in 1948 (March to August) reporting on numerical experiments.

He uses both the term “fictitious” and “mock” to describe the term, But not “artificial”. All of these are unfortunate in their connotation.

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial m}\left(p+q\right)=0 \rightarrow \frac{\partial u}{\partial t}+\frac{\partial}{\partial m}\left(p-\mu\frac{\partial u}{\partial x}\right)=0$$

The VNR50 paper discusses “q” which has led to a Hugoniot focused view point while R48 originally discussed the viscosity, which is a very different view point.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Physics - general

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Classification changed

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Per J. Brown, FSS-16 Date: 8-26-93
By Marlene Lujan, CIC-14 Date: 8-8-96

LA-699

Series A

[redacted]

19 August 1948 This document contains 33 pages.

PROPOSED NUMERICAL METHOD FOR CALCULATION OF SHOCKS, II

Work done by:
Robert D. Richtmyer

Report written by:
Robert D. Richtmyer

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What is the point of this in the present context?

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- **Von Neumann-Richtmyer (VNR) form was subtly different than Richtmyer's original formulation.**
- **This has put a focus on the Hugoniot (shock) locus rather than other shock descriptions.**
  - The Richtmyer form focuses on the viscous form, and the Rayleigh line, which may rid solutions of anomalies that have plagued computations for decades.
  - It also allows for a more systematic framework to develop automatic coefficient prescriptions for the viscosity.
- **The same approach can work with other dissipation approaches, i.e., Riemann solvers**
  - using the Rayleigh line instead of the Hugoniot.

$$c = \sqrt{-\frac{1}{\rho^2}\frac{\Delta p}{\Delta V}}$$

Steepening of Shock Waves

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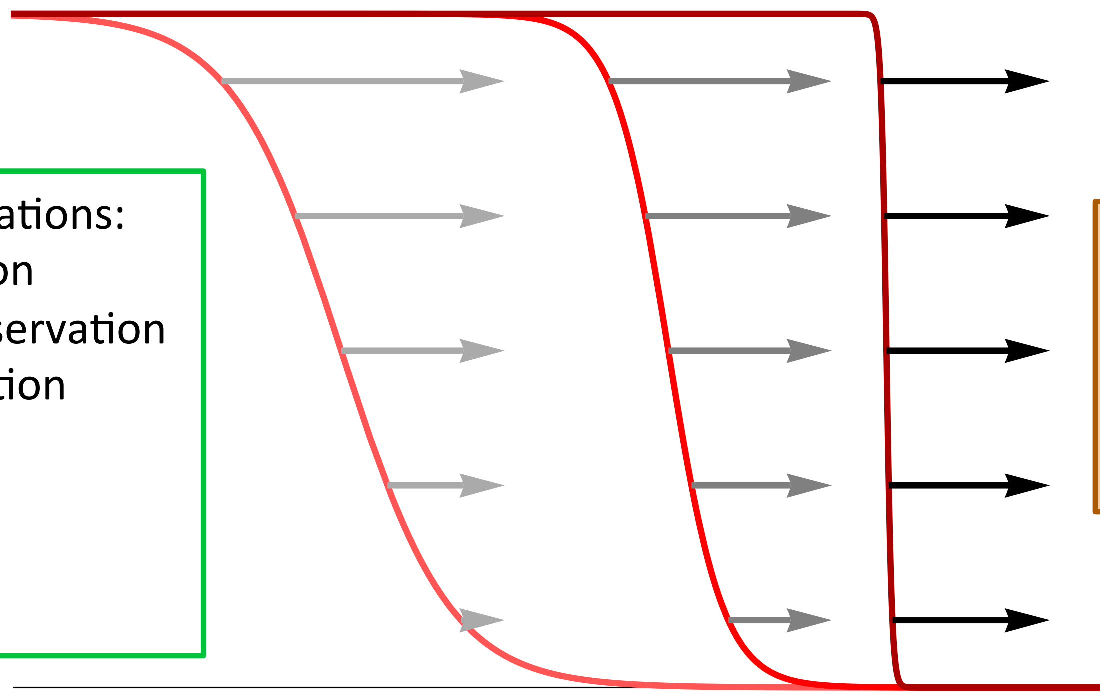
High pressure part of wave profile moves faster than low pressure part -> steepening into step shock.

Rankine-Hugoniot equations:
- Mass conservation
- Momentum conservation
- Energy conservation

Materials relations:
- Incomplete EOS: P(V,E)

Hydro-dynamic equations:
- Mass conservation
- Momentum conservation
- Energy conservation

Materials relations:
- Viscosity
- Heat conduction
- EOS: E(V,S)



Real materials properties give shocks too thin to resolve in hydro-codes.

Thickness and details of shock profile are determined by viscosity and heat conduction but is neglected in the RH equations.

Artificially stopping steepening of Shock Waves

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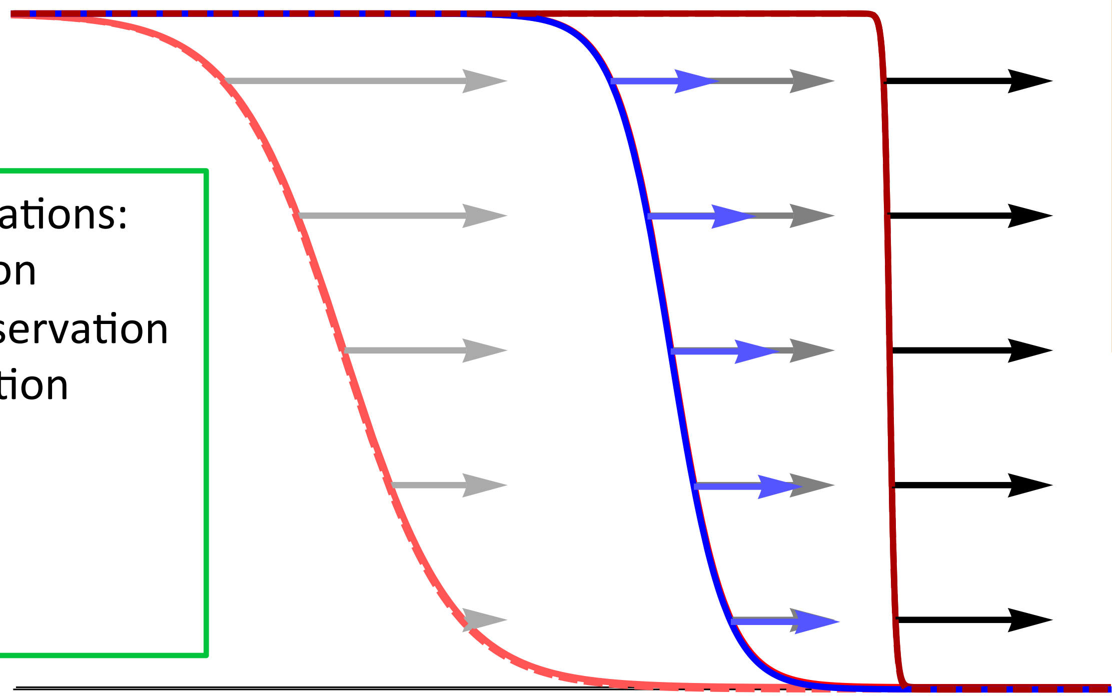
Cannot be described by the Rankine-Hugoniot Equations

Hydro-dynamic equations:
- Mass conservation
- Momentum conservation
- Energy conservation

Materials relations:
- Viscosity
- Heat conduction
- EOS: E(V,S)



R48 explicitly discusses desiring a steady wave profile in the shock frame.

We need to tune the materials properties artificially to make the full profile move at the same speed.

Steady Wave Equations

$x = z - U_s\, t$ in hydro equations:

$$\mu \frac{du}{dx} = -q = \left[(P - P_0) + m^2 (V - V_0)\right]$$

$$\frac{\kappa}{m}\frac{dT}{dx} = \left[(E - E_0) - (V_0 - V)(P_0 + \tfrac{1}{2} m^2 (V_0 - V))\right]$$

μ is the viscosity
κ is the heat conductance
m is the constant mass flux
u is the particle velocity

Richtmyer derived his original expression for artificial viscosity from these equations (with $\kappa=0$) in his 1948 LA-671 report.

If μ and κ are very small we recover the Rankine-Hugoniot equations.

Initial and final states are invariant in time and space, thus are connected by the Rankine-Hugoniot equations.

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The profile states are not necessarily shock states.

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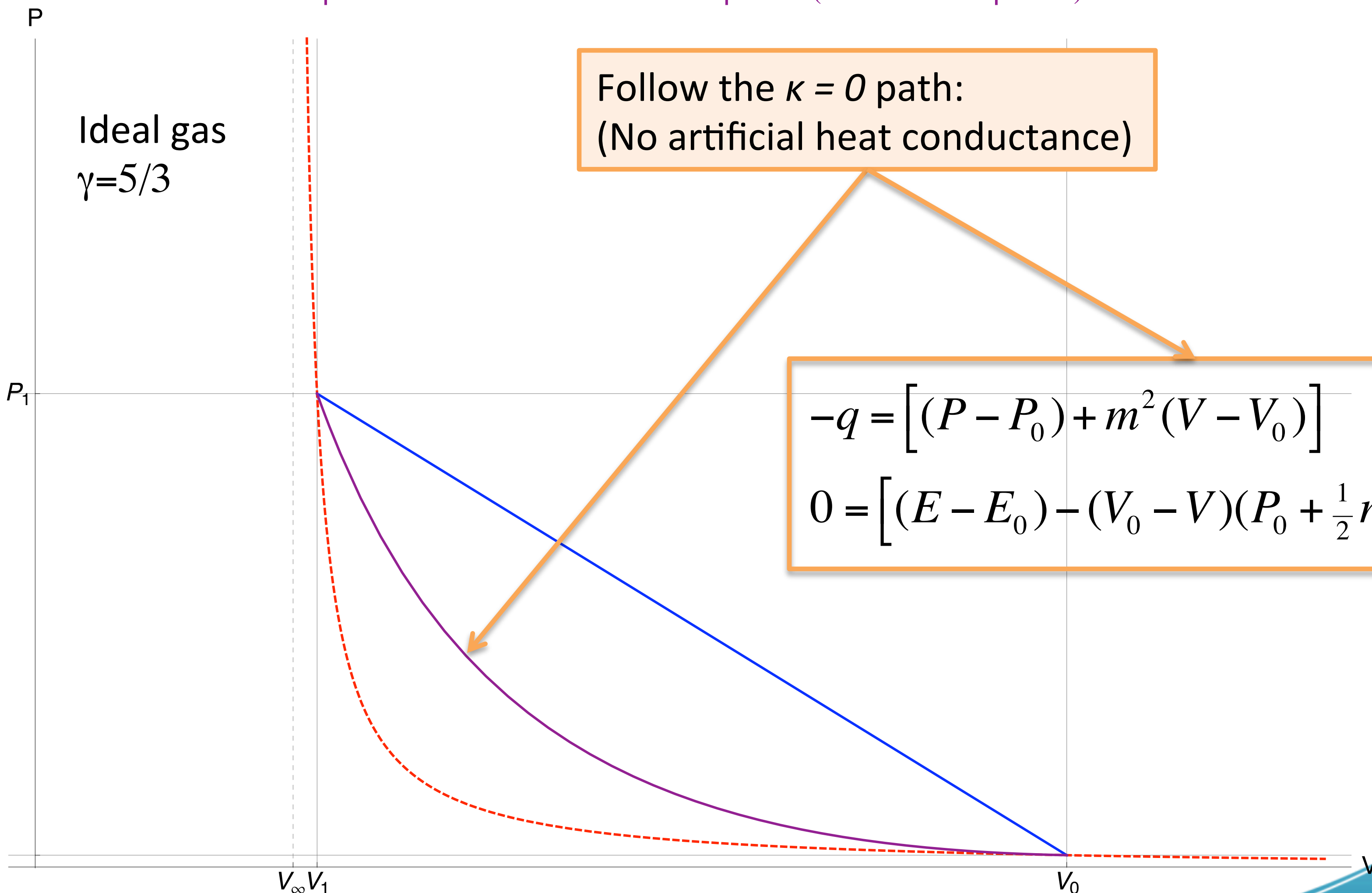
Shock profile states

Red dashed: Hugoniot

Blue: Rayleigh line (step profile)

Purple: No heat conduction within profile (smeared out profile)



$$-q = \left[(P - P_0) + m^2 (V - V_0)\right]$$

$$0 = \left[(E - E_0) - (V_0 - V)(P_0 + \tfrac{1}{2} m^2 (V_0 - V))\right]$$

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Viscosity, no heat conductance:

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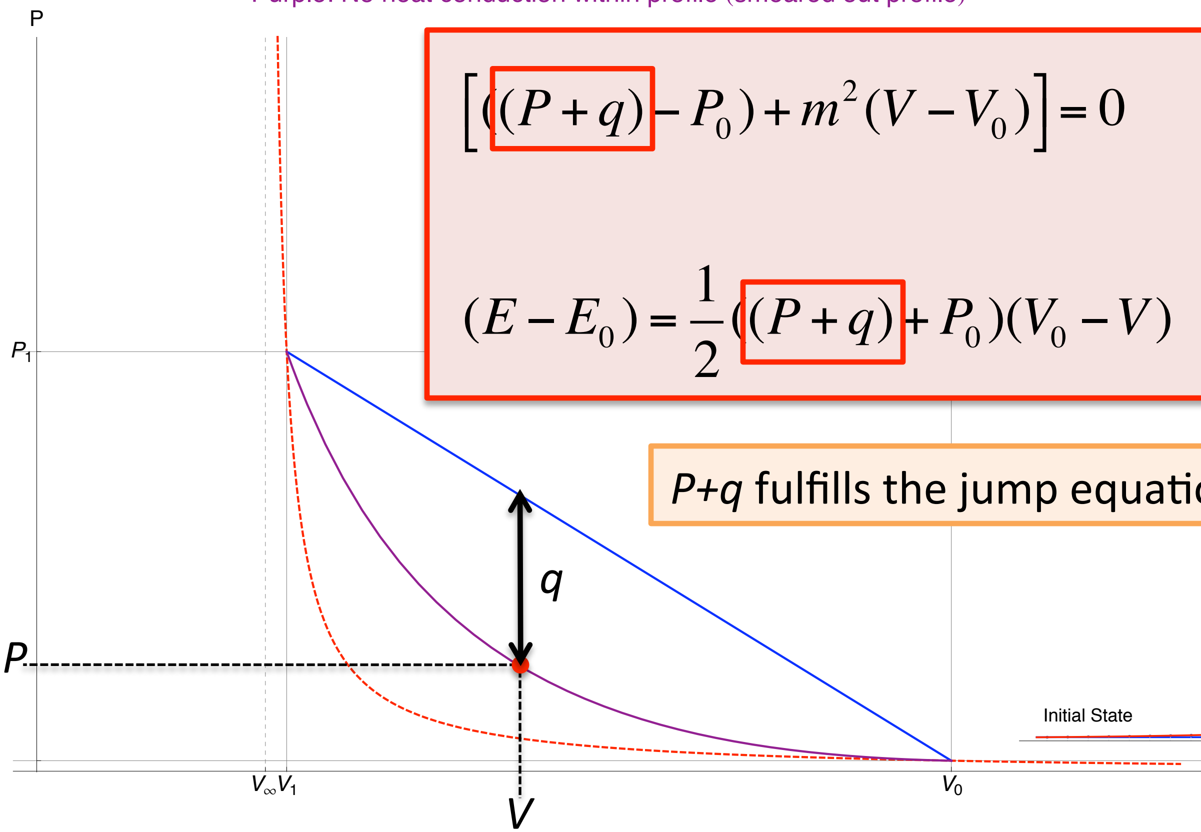
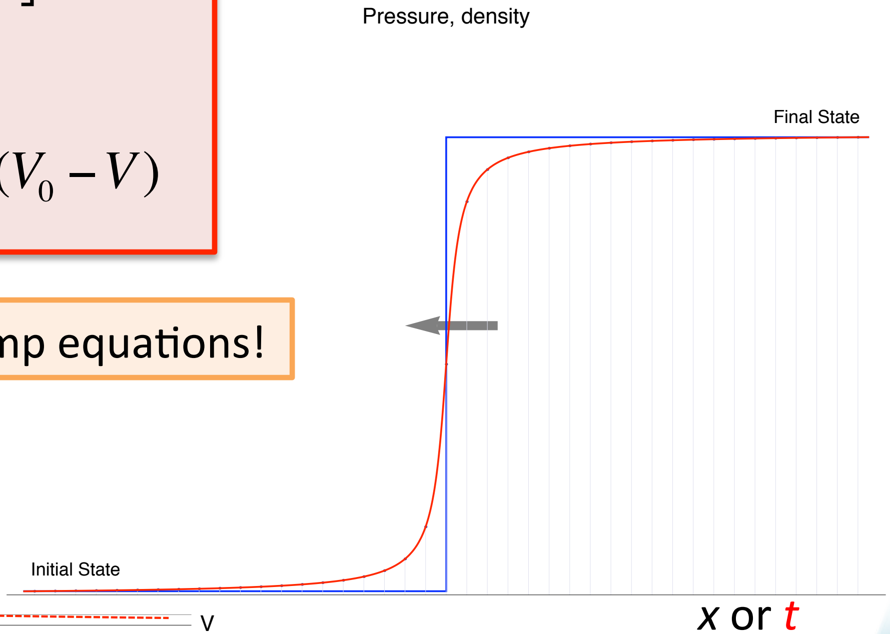


$$\left[\left(\boxed{(P+q)}-P_0\right)+m^2(V-V_0)\right]=0$$

$$(E-E_0)=\frac{1}{2}\left(\boxed{(P+q)}+P_0\right)(V_0-V)$$

P+q fulfills the jump equations!



Makes sense: Viscosity is simply a delay, the state in a fixed point does not go directly from (P_0,V_0) to (P_1,V_1). However, eventually it gets there, faster if q is smaller (directly if $q=0$).

Now we know what value *q* must have. How do we implement it?

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1) $-q = \left[(P - P_0) - m^2 (V_0 - V) \right]$

2) $-q = \left[(E - E_0) - \frac{1}{2}(P + P_0)(V_0 - V) \right]$

3) $-q = \frac{V}{\Gamma}\left(\gamma \frac{P}{V} + \frac{dP}{dV} \right) = \frac{V}{\Gamma}\left(-\left.\frac{\partial P}{\partial V}\right|_S + \frac{dP}{dV} \right)$

4) $-q = T\frac{dS}{dV}$

The equation for *q* can be written in many ways, which is best to implement?

- Can we get *m*, the mass flux, accurately?
- Storing the initial state?
- Calculate *dP/dV* as *ΔP/ΔV*?
- Calculate *dS/dV* as *ΔS/ΔV*, but how do we get *ΔS?*
- Use weak shock approximations for entropy?
- …

Probably need to be combined with limiters.
This *q* gives a steady wave, but is the profile wide enough? We do NOT want to have artificial heat conductance. Can we use the limiters cleverly in this case?

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In 1942 Hans Bethe derived the entropy production in a weak shock:

$$T\Delta S = -\tfrac{1}{6} G c^2 \left(\frac{\Delta V}{V}\right)^3 \rightarrow T\Delta S = -\tfrac{1}{6}\frac{G}{c}(\Delta u)^3$$

Where G is the fundamental derivative:

$$G = -\frac{1}{2} V \frac{\partial^2 P / \partial V^2 \big|_S}{\partial P / \partial V \big|_S} = \frac{1}{c}\frac{\partial \rho c}{\partial \rho}\bigg|_S$$

$$(\rho c)^2 = -\partial P / \partial V \big|_S$$

and we have used

$$\rho c \Delta V = \Delta u \rightarrow \frac{\Delta V}{V} = \frac{\Delta u}{c}; c \propto C \frac{\Delta x}{\Delta t}$$

And we arrive at the main formula for the artificial viscosity that we have tested so far:

$$q = -T\frac{dS}{dV} \approx -T\frac{\Delta S}{\Delta V} = \frac{1}{6}\frac{G}{c}(\Delta u)^3 \frac{\rho c}{\Delta u} = \frac{1}{6}\rho G (\Delta u)^2$$

Note: This is quadratic but with a 'new' coefficient.

Turning this into a practical algorithm

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- The standard form still works as a default and foundation.
- We add several elements to this form to achieve utility
  - Automatic quadratic coefficient calculation, two forms
  - Viscosity based on the local Rayleigh line
  - Bounding selection criteria
  - Proper coefficient of linear viscosity
  - A "smoothed" shock switch
- We also introduce changes to the viscosity based on the finite width of the shock.
- Together with limiters and hyperviscosity it works well

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How to take the Rayleigh line equation and derive a quadratic viscosity?

- **The key is to approximate the Rayleigh line locally,**

$$p - p_0 + m^2\left(V - V_0\right) = \mu \frac{\partial u}{\partial x} \rightarrow \Delta p + m^2 \Delta V = \mu \frac{\partial u}{\partial x}$$

- **Putting this into the viscous form of the viscosity:**

$$-\mu \frac{du}{dx} = \frac{1}{6}\rho G (\Delta u)^2 \Rightarrow$$

  – Rearrange and approximate with local data, solve for the viscous coefficient

$$\Delta_j p = p_{j+1} - p_{j-1} \qquad \Delta_j V = 1/\rho_{j+1} - 1/\rho_{j-1} \qquad m_j = \rho_j c_j$$

- **This produces an estimate for the viscosity through the estimate of an effective quadratic coefficient,**

$$\frac{1}{6}G \approx \tilde{G}_j = \left| \frac{\left(\Delta_j p + m_j^2 \Delta_j V\right)}{\rho_j m_j \Delta_j V \Delta_j u} \right|$$

or

$$\tilde{G}_j = \left| \frac{\left(\Delta_j p + m_j^2 \Delta_j V\right)}{\rho_j m_j^2 \Delta_j V \Delta_j V} \right|$$

Automatic quadratic coefficient

- **From the standard Hugoniot-based selection**

$$p_s - p_0 = \rho_0\left(c_0 + c_2\Delta u\right)\Delta u = \rho_0\left(c_0 + \tfrac{1}{2}\mathrm{G}\Delta u\right)\Delta u$$

$$\mathrm{G}_j = \frac{1}{c}\frac{\partial \rho c}{\partial \rho}\bigg|_S \approx \frac{1}{c_j}\left(\frac{\rho_{j+1}c_{j+1} - \rho_{j-1}c_{j-1}}{\rho_{j+1} - \rho_{j-1}}\right)$$

- **This can be shown to be equivalent to the quadratic viscosity introduced by Lax & Wendroff in 1960**

$$\rho_j \mathrm{G}_j \left|\frac{\partial u}{\partial x}\right| \frac{\partial u}{\partial x} \approx \left|\frac{\partial \rho c}{\partial x}\right| \frac{\partial u}{\partial x}$$

Just an aside

- **Finally we bound the selection to assure stability and guard against noise contaminating the solution**
  - The analytical value of G is used based on the shock entropy production in the weak shock limit. (W>1 measures the width of the shock)

$$c_2 = \mathrm{median}\left(\tfrac{1}{6}\mathrm{G}, \hat{\mathrm{G}}_j, c_{upper}\right) \quad \hat{\mathrm{G}}_j = \min\left(\tfrac{1}{6}W\mathrm{G}_j, \tilde{\mathrm{G}}_j\right)$$

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Computing W_j (Using the discrete profile)

- We compute viscosity usually developing the coefficients by considering an ideal shock jump.
- The real captured shock is discrete containing several points.
- A quadratic viscosity is nonlinear (second-order) in the size of the jump
  - Example, a jump of 2 across two cells produces a substantial difference in the jump size squared.
- Entropy is third order in the size of the jump, has a larger impact.
- The wider the discrete shock, the larger the impact.

Examples plotted as a function of the symmetry of the jump and number of cells

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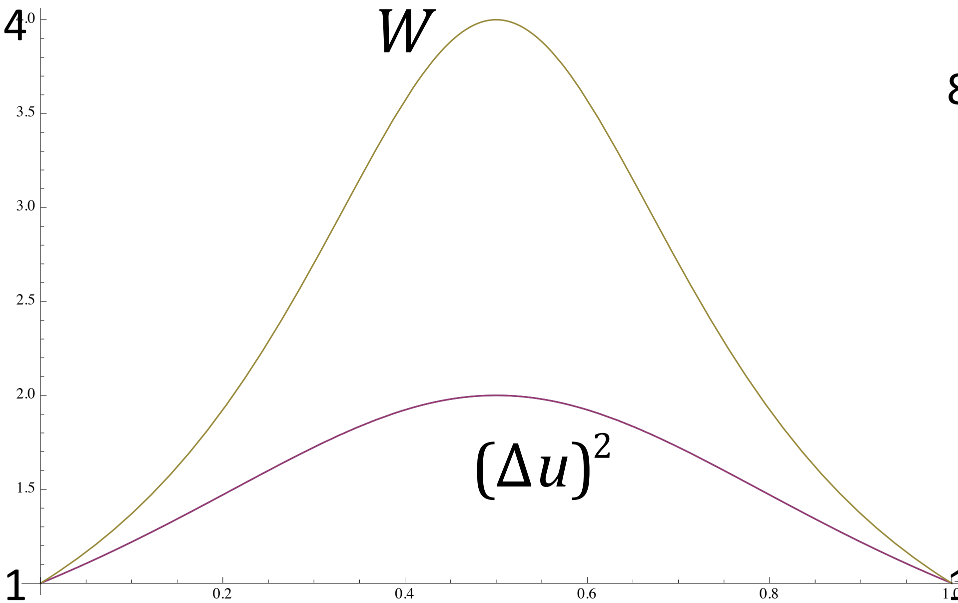

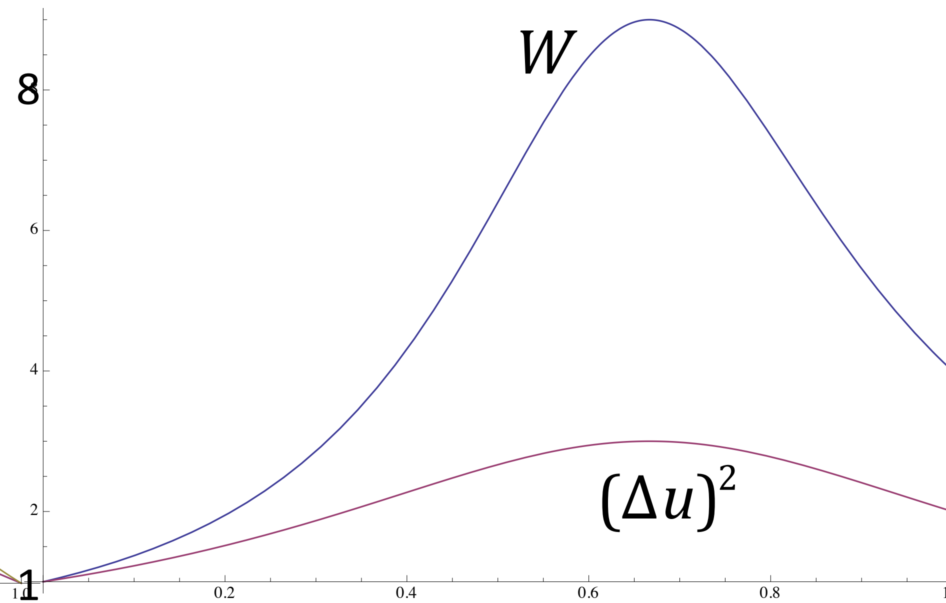

Multiplier on the coefficient
For a two cell wide shock jump

$$W_j = \left(V_{j+1} - V_{j-1}\right)^3 \Big/ \left[\left(V_{j+1} - V_j\right)^3 + \left(V_j - V_{j-1}\right)^3\right]$$

The upper curve is "W" and the lower curve is the ratio of jump sizes.

Multiplier on the coefficient
For a three cell wide shock jump,
2 steps the same (pre and post jump).

The Riemann solver analogy fills in the final details with the correct linear coefficient

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- **The usual Hugoniot-based analysis sets this as equal to "1"**
  - This analysis is completely correct, but the interpretation is wrong due to a simple error.
- **Looking at this selection in the context of a first-order method with a linearized Riemann solver clears up the confusion.**
  - The second term is the linear dissipation

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{m_j}\left(p_{j+1/2} - p_{j-1/2}\right) \quad p_{j+1/2} = \frac{1}{2}\left(p_j^n + p_{j+1}^n\right) - \frac{\overline{\rho c}}{2}\left(u_{j+1}^n - u_j^n\right)$$

- **Computing the standard fluxes gives the right coefficient scale… everything is half as big as the usual analysis indicates. C_1=1/2, and C_2 is half the size you thought it should be**
  - (consistent with Morgan's latest work – 2013).

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A small change to the hyperviscosity

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- **Ed Love recognized that our hyperviscosity form did not necessarily satisfy the second law.**
  - This results by the difference in the sign of the hyperviscous term and the deformation rate,

    $Q_{total} = Q(d) + c_4 \left(Q(d) - Q(\bar{d}) \right)$
- **Normally the viscosity satisfies the 2nd law** $Q(d) \cdot d \leq 0$
- **This can be cured by a relatively simple change to the original form**

  $Q_{total} = Q(d) + c_4 \operatorname{sign}\left(Q(d) \right) \left| Q(d) - Q(\bar{d}) \right|$
  - Satisfaction of the second law is recovered.
  - Practically this is a small change only impacting results near severely rough solution.

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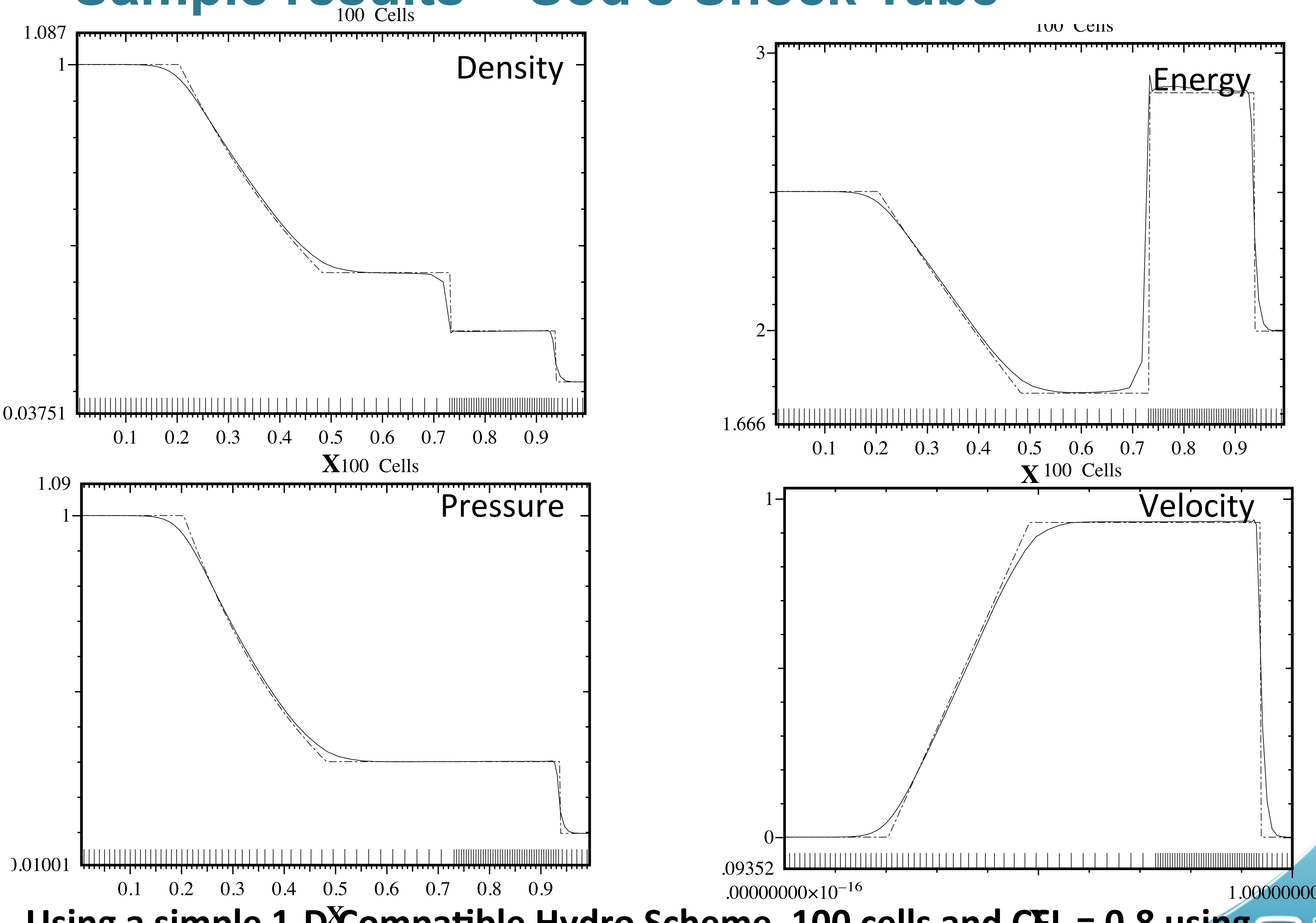
Sample results – Sod's Shock Tube



Using a simple 1-D Compatible Hydro Scheme, 100 cells and CFL = 0.8 using the new viscosity formulation. Sod's shock tube (170 cycles)

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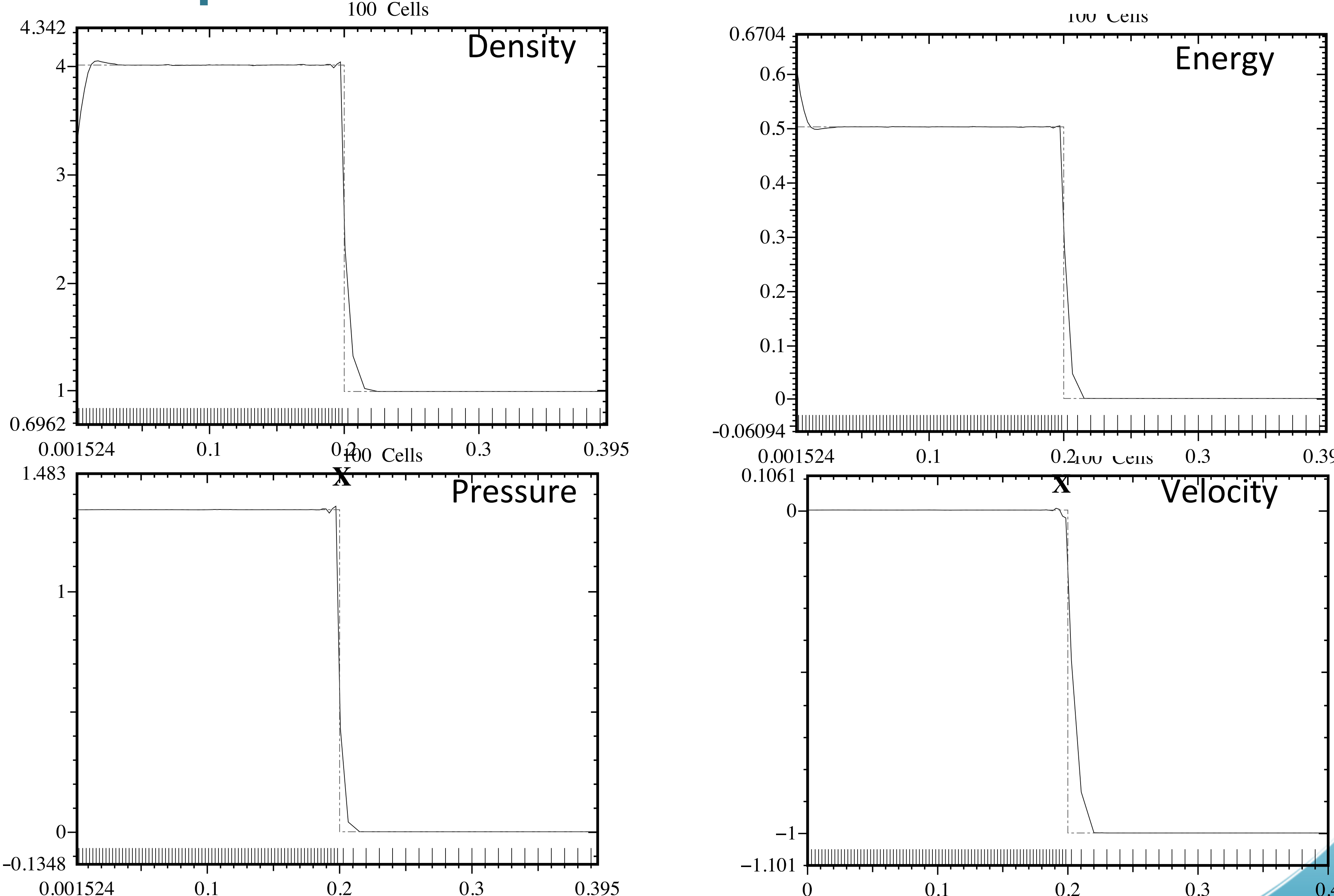
Sample results – Noh’s Problem



Using a simple 1-D Compatible Hydro Scheme, 100 cells and CFL = 0.8 using the new viscosity formulation. Planar Noh’s problem (400 cycles)

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Conclusions

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- **The differences between the VNR form and the original Richtmyer form are subtle, but important.**
  - The focus on the viscous form opens the door to looking at the Rayleigh line instead of the Hugoniot.
  - Neither perspective is correct, and the differences are instructive and need to be integrated into the numerics
- **None of the coefficients are "arbitrary"**
- **Old papers are full of riches waiting to be (re)discovered.**
- **Getting the coefficients correct is a substantial change**
- **Results are encouraging**

A few final thoughts

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- **It is probably useful to consider the imposition of a discrete entropy condition associated with the artificial viscosity**
  - To get convergence to the correct weak solution the entropy condition is important.
  - This has not been used for Q's. It has for Riemann solutions
- **The quadratic coefficient is connected to nonlinear stability (CD Munz 1994, Caramana & Shashkov 1998)**
  - Inhibits element inversion due to over-compression.