A Symmetry Preserving Dissipative Artficial Viscosity in r-z Geometry

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Motivation and Purpose

In the staggered r-z discretization, develop an artificial viscosity that

- **is genuinely r-z (not area-weighted)**
  - **preserves spherical symmetry**
    - **and is strictly dissipative**

Outline of This Presentation

- **Intro: Area-Weighted schemes**
- **Our viscous force** (LapEdge)**: general form and properties**
- **Equiangular polar grid**
  - **Velocity**
  - **Internal energy**
  - **Boundary: velocity and total energy**
- **How does it work on general grids**
  - **Chord length**
  - **Generally applicable BC at the z-axis**
- **Numerical results**

Area-Weighted Schemes

- Using Cartesian pressure and viscous forces, the momentum equation reads

$$m_p \frac{d\mathbf{U}_p}{dt} = r_p \sum_{c(p)} (\mathbf{F}^{D}_{pc} + \mathbf{F}^{AD}_{pc}).$$

- Cell mass m_c taken as fundamental. The cylindrical nodal mass m_p defined so that $\frac{m_p}{r_p}$ is independent of angle and time invariant for a symmetric grid $\Rightarrow$ spher. symmetry preserved
- Internal energy change is typically based on

$$m_c \frac{d\,\varepsilon_c}{dt} = -\sum_{p(c)} r_p \,(\mathbf{F}^{D}_{pc} + \mathbf{F}^{AD}_{pc}) \cdot \mathbf{U}_p$$

- Contribution to the internal energy from the viscous term is not dissipative, i.e.

$$\sum_{p(c)} r_p \,\mathbf{F}^{AD}_{pc} \cdot \mathbf{U}_p \leq 0$$

  is not necessarily true. This is the case for the tensor viscosity of CS.
- By CS we refer to the original method [Campbell and Shashkov, JCP 172 (2001)]. For other formulations see [Wendroff, JCP 229 (2010)] or [Kolev and Rieben, JCP 228 (2009)], for an improved version see [Lipnikov and Shashkov, JCP 229 (2010)].

The LapEdge Viscous Force - General Form

- We are going to define a genuinely cylindrical (not area-weighted) artificial viscous force, so that the equation for velocity becomes

$$m_p \frac{d\mathbf{U}_p}{dt} = \sum_{c(p)} (r_p \, \mathbf{F}_{pc}^D + \mathbf{F}_{pc}^A).$$

- This force will be a sum of edge forces, namely,

$$\mathbf{F}_{pc}^A = \sum_{e(p,c)} \mathbf{f}_{pe},$$

each of which will have the form

$$(\mathbf{f}_{pe})_r = \sigma_{ce} \left(r_e \, \Delta u_{pe} - a_e \frac{u_e}{r_e} \right), \qquad (\mathbf{f}_{pe})_z = \sigma_{ce} \, r_e \, \Delta v_{pe}.$$

- Motivated by the form of the Laplacian of a vector in r-z geometry:

$$L(\mathbf{U})_r = \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial z} \left(r \frac{\partial u}{\partial z} \right) - \frac{u}{r} \right], \qquad L(\mathbf{U})_z = \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{\partial}{\partial z} \left(r \frac{\partial v}{\partial z} \right) \right].$$

The LapEdge Viscous Force - Details

$$\boxed{(\mathbf{f}_{pe})_r = \sigma_{ce} \left(r_e \, \Delta u_{pe} - a_e \, \frac{u_e}{r_e} \right), \qquad (\mathbf{f}_{pe})_z = \sigma_{ce} \, r_e \, \Delta v_{pe}}$$

- **r_e is the arithmetic average r of the edge**
- **u_e is the average of the radial components of the endpoint velocities**
- **$\sigma_{ce} \geq 0$ supplies the proper dimensions (density times velocity times length)**
  The exact form of σ_{ce} critically affects the behavior of our method, but it plays no role in the symmetry and conservation properties of the scheme. Here we use in particular

$$\sigma_{ce} = S_c \, \ell_{ec}^{\text{char}} \, \kappa_c$$

  - S_c is the switch to turn off viscosity in expanding cells (detected by velocity divergence).
  - κ_c is the viscosity module $\kappa_c = \rho_c \left(k_2 \frac{\Gamma_c+1}{4} \Delta U_c + \sqrt{\left(k_2 \frac{\Gamma_c+1}{4} \Delta U_c \right)^2 + (k_1 \, s_c)^2} \right)$
  - ℓ_{ec}^{char} is the characteristic length $\ell_{ec}^{\text{char}} = \frac{A_c}{4 \, \ell_e}$, where A_c is the area (Cartesian volume) of cell c and ℓ_e is the length of edge e.

- **The constant $a_e \geq 0$ will be constructed so that if the grid is equi-angular polar then symmetry will be preserved. a_e will depend only on a few neighbors of p.**

The LapEdge Viscous Force - Properties

$$(\mathbf{f}_{pe})_r = \sigma_{ce} \left(r_e \, \Delta u_{pe} - a_e \, \frac{u_e}{r_e} \right), \qquad (\mathbf{f}_{pe})_z = \sigma_{ce} \, r_e \, \Delta v_{pe}$$

If the edge $e(p,c)$ connects point p of c to point q of c, then the velocity difference $\Delta \mathbf{U}_{pe}$ is

$$\Delta \mathbf{U}_{pe} = \left(\Delta u_{pe} \, , \, \Delta v_{pe} \right) = \mathbf{U}_q - \mathbf{U}_p .$$

- **Then clearly z-momentum is conserved, since the z-component of the force at p is the negative of it at q.**
- **It is easy to see that this viscous force is dissipative: Since**

$$[\mathbf{U}_q - \mathbf{U}_p] \cdot \mathbf{U}_p + [\mathbf{U}_p - \mathbf{U}_q] \cdot \mathbf{U}_q = -\|\mathbf{U}_q - \mathbf{U}_p\|^2,$$

the internal energy change is

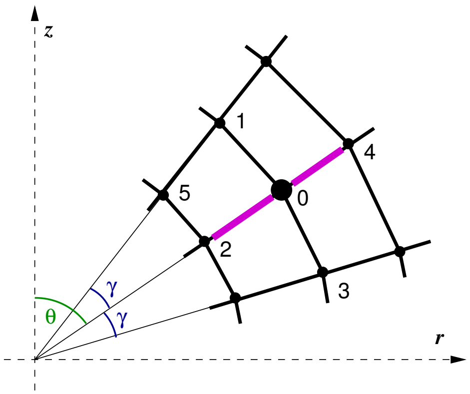
$$\sum_{p(c)} \left(\mathbf{F}^A_{pc} \cdot \mathbf{U}_p \right) = -\sum_{e(c)} \sigma_{ce} \left(r_e \, \|\mathbf{U}_q - \mathbf{U}_p\|^2 + 2 \, a_e \, \frac{u_e^2}{r_e} \right) \le 0.$$

Equiangular Polar Grid - Acceleration

Assume

$$r_{i,j} = R_j \sin(i\gamma), \qquad z_{i,j} = R_j \cos(i\gamma),$$

so that both the spherical radii R_j and the angular interval γ are known.



- **Assume symmetric data, including the σ coefficients. Without losing generality, we set all $\sigma_{ce} = 1$**
- **The points** 1, 0, **and** 3 **lie on the circle of radius R_0, while the points** 2, 0, **and** 4 **lie on the ray with angle θ.**

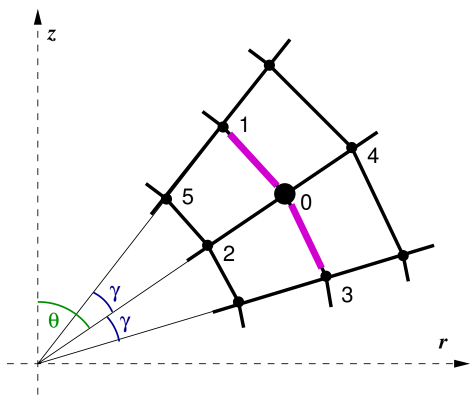
From "rays" (0,2) and (0,4):

- **Consider first the viscous force contribution from the edges** (0,2) **and** (0,4)**. For all such edges forming a straight line we set $a_e = 0$.**
- **Velocity field is symmetric $\Rightarrow \mathbf{U}_2$, $\mathbf{U}_0$, and $\mathbf{U}_4$ are directed radially** (all inward or all outward) **with magnitudes independent of angle θ. That is, each of those velocities is of the form** $\mathbf{U}_k = \pm \|\mathbf{U}_k\| \, (\sin\theta, \cos\theta)$ **for $k \in \{0, 2, 4\}$, with $\|\mathbf{U}_k\|$ independent of θ.**
- **Each $r_k = R_k \sin\theta \Rightarrow$ the viscous force at point** 0 **from these edges has the form**

$$\mathbf{f}_{0,\text{rays}} = h \sin\theta \, (\sin\theta, \cos\theta), \qquad h \ \textbf{independent of angle } \theta,$$

that is, it is directed radially and the components have $\sin\theta$ as a common factor.

Equiangular Polar Grid - Acceleration



From "circles" (0,1) and (0,3):

- **Symmetry $\Rightarrow$ safely assume that the velocity vectors at 1, 0, and 3 are radial unit vectors. Thus**

$$u_1 = \sin(\theta - \gamma), \qquad u_0 = \sin\theta, \qquad u_3 = \sin(\theta + \gamma),$$
$$v_1 = \cos(\theta - \gamma) \qquad v_0 = \cos\theta \qquad v_3 = \cos(\theta + \gamma).$$

- **The r averages are** $\frac{1}{2}\,(r_{1,3} + r_0) = \frac{1}{2}\,R_0(\sin\theta + \sin(\theta \mp \gamma))$.

- **Components of the viscous force at 0 reduce to**

$$(\mathbf{f}_{0,\text{circ}})_r = \cdots = -2\,R_0 \sin^2\gamma \sin\theta \sin\theta + R_0 \sin^2\gamma - 2\,a/R_0,$$
$$(\mathbf{f}_{0,\text{circ}})_z = \cdots = -2\,R_0 \sin^2\gamma \sin\theta \cos\theta.$$

- **Therefore, if we set** $a = \frac{1}{2}\,R_0^2 \sin^2\gamma$, **this force is radial.**

- **Common factor $\sin\theta \Rightarrow$ the acceleration will be radial and independent of angle θ.**

- **Thus we define** $a_e = \begin{cases} 0 & \textbf{on rays} \\ \frac{1}{2}\,R_j^2 \sin^2\gamma & \textbf{on a circle of radius } R_j. \end{cases}$

Equiangular Polar Grid - Internal Energy

- **The internal energy equation using our edge viscosity is now**

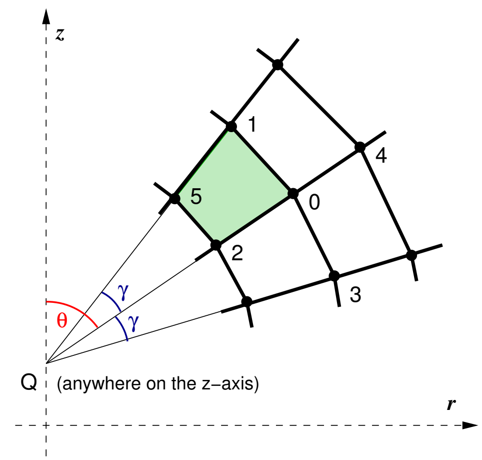
$$m_c \frac{d\,\varepsilon_c}{dt} = -\sum\nolimits_{p(c)} (r_p \mathbf{F}^D_{pc} + \mathbf{F}^A_{pc}) \cdot \mathbf{U}_p,$$

and thus symmetry is maintained if $\delta\varepsilon_c \equiv \frac{1}{m_c} \sum_{p(c)} \mathbf{F}^A_{pc} \cdot \mathbf{U}_p$ **is a function only of** R

- **Consider the cell with vertexes** (1,5,2,0) **and its cylindr. volume** V
- Let Δ_1 be the planar area of the triangle (1,Q,0) and
  Let Δ_2 be the planar area of the triangle (5,Q,2). Then

$$V = \tfrac{1}{3}[(r_1 + r_0)\Delta_1 - (r_5 + r_2)\Delta_2].$$

Note that Δ_1 and Δ_2 are independent of angle θ.



- **It can be shown that for this cell,**

$$\sum\nolimits_{p(c)} \left(\mathbf{F}^A_{pc} \cdot \mathbf{U}_p\right) = A(r_1 + r_0) + B(r_5 + r_2),$$

where A and B are again independent of angle θ.

- **Then**

$$\frac{1}{3}\,\rho_c\,\delta\varepsilon_c = \frac{A(r_1 + r_0) + B(r_5 + r_2)}{(r_1 + r_0)\Delta_1 - (r_5 + r_2)\Delta_2} = \frac{A + B\,\frac{r_5+r_2}{r_1+r_0}}{\Delta_1 - \frac{r_5+r_2}{r_1+r_0}\,\Delta_2},$$

but $\frac{r_5+r_2}{r_1+r_0}$ is independent of angle θ, and therefore so is $\delta\varepsilon_c$.

Equiangular Polar Grid - BC for Velocity

- **Clearly, points on the z-axis (r=0) require special treatment**
- **Suppose full symmetry and consider nodes on the same spherical radius R**
- **For interior nodes, the contribution to the *viscous acceleration* is**

$$\frac{d\delta \mathbf{U}_p}{dt} = \frac{\mathbf{F}_p^A}{m_p} = \frac{h \sin\theta_p (\sin\theta_p, \cos\theta_p)}{\beta \sin\theta_p} = \frac{h}{\beta}(\sin\theta_p, \cos\theta_p)$$

  with h and β independent of angle θ
- **To preserve symmetry, it seems inevitable to take at the boundary, i.e. at $(r, z)_q = (0, R)$,**

$$\frac{d\delta u_q}{dt} = 0, \qquad \frac{d\delta v_q}{dt} = \frac{h}{\beta}.$$

  But there is a simple generally applicable BC that preserves symmetry, which we show later.
- **However it is interesting to see what h is:**
  In logically rectangular notation $(r, z)_{i,j} = R_j(\sin\theta_i, \cos\theta_i)$, $(u, v)_{i,j} = g_j(\sin\theta_i, \cos\theta_i)$.
  Assuming the interior node force in the form $\mathbf{F}_p^A(u, v)_{i,j} = h\ \sin\theta_i\ (\sin\theta_i, \cos\theta_i)$ yields

$$h = \left(\underbrace{\tfrac{1}{2}\left[(R_{j+1} + R_j)(g_{j+1} - g_j) + (R_j + R_{j-1})(g_{j-1} - g_j)\right]}_{\text{from rays}} \underbrace{- 2\, g_j\, R_j\ \sin^2\gamma}_{\text{from circles}} \right).$$

- **Note that the momentum equation at bdry $\beta \frac{d\delta v_q}{dt} = h$ is consistent with the AW mom. eqs.**

Equiangular Polar Grid - Total Energy

- So far proved: **our force preserves int. energy symmetry for all cells, incl. those at the z-axis.**
- **However, we obtained boundary node acceleration by ratio $\mathbf{F}/m$, without choosing $\mathbf{F}$ or m.** **The boundary accel. is not defined by our force $\mathbf{F}$. This affects the tot. ener. conservation**
- **Introduce mass m_{pc} such that** $m_c = \sum_{p(c)} m_{pc}$, $m_p = \sum_{c(p)} m_{pc}$ **and** $\frac{dm_{pc}}{dt} = 0$.

■ **For points not on the z-axis there is a nodal flux function G_{pc}, that is, $\sum_{c(p)} G_{pc} = 0$**

■ **For cells away from z-axis the tot. ener. change is $\frac{dE_c}{dt} = \sum_{p(c)} G_{pc}$, so that for those cells**

$$\sum_c \frac{dE_c}{dt} = \sum_c \sum_{p(c)} G_{pc} = 0 \qquad \checkmark \qquad (1)$$

■ **For cells at the z-axis, the proof of (1) involves a division by m_p at points for which $r_p = 0$.**

■■ **For pure AW using m_{pc}: both m_{pc} and $\mathbf{F}_{pc}$ are zero if p is a point on the z-axis**

$\Rightarrow$ for finite acceleration the boundary nodes are not present in the proof of (1)

$\Rightarrow$ cells at the z-axis can be included in the conservation of total energy. $\checkmark$

■■ **Our $\mathbf{F}_{pc}$ is defined at all nodes, including those on the z-axis.**

◇ To guarantee symmetric $\delta\varepsilon_c$ and $\delta\varepsilon_c \geq 0$ for cells at z-axis, the "work" of the z-axis viscous force on the boundary nodes must be included in the internal energy change of those cells.

◇ But the viscous contribution to the acceleration of the z-axis nodes is obtained by the constraint of symmetry preservation, not from $\mathbf{F}/m_p$ (as it would have to be for energy conservation up to the boundary).

General Grids

- **Up to now we assumed spherically symmetric initial data and grid**
- **Now we consider a general logically rectangular grid with only one restriction: the z-axis must be one of the grid curves, that is either i=constant (i-curve) or j=constant (j-curve).** (We know of no other scheme where this is not the case.)
- **Suppose it is an i-curve. Then if there are circles and rays the rays must be i-curves.**

Thus, we take $a_e = 0$ on all i-curves

Now we know that only the j-curves are possibly circles.

- **Returning to the equiangular polar case , we can prove that**

$$\tfrac{1}{2}R_j^2 \sin^2\gamma = \tfrac{1}{8}\|\mathbf{X}_{i+1,j} - \mathbf{X}_{i-1,j}\|^2, \quad \text{where } \mathbf{X}_{i,j} = (r,z)_{i,j} = R_j\big(\sin(i\gamma), \cos(i\gamma)\big).$$

- **Therefore, as the general default value of a_e for the edge ($i+\frac{1}{2}, j$) connecting the point at (i,j) to the point at $(i+1,j)$ we propose**

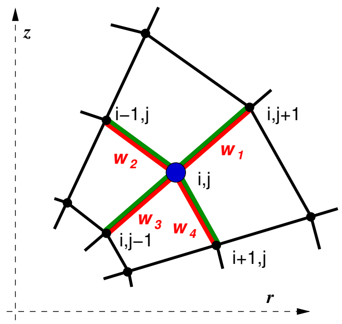
$$a_{i+1/2,j} = \frac{1}{16}\left(\|\mathbf{X}_{i+1,j} - \mathbf{X}_{i-1,j}\|^2 + \|\mathbf{X}_{i+2,j} - \mathbf{X}_{i,j}\|^2\right) \quad \text{on all } j\text{-curves}.$$

If there are equi-angular polar circles, the above will find them.
If not, then the default provides an acceptable viscosity.

General Grids - z-axis Boundary Condition

- **Suppose that $\sigma = 1$. Consider an equi-angular polar grid with symmetric velocity**
- **For an interior node (i, j), $i > 0$ we can write $\frac{d\delta v_{i,j}}{dt}$ as**

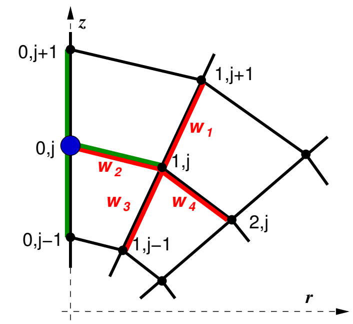
$$\underbrace{\frac{r_{i,j+1} + r_{i,j}}{2\, m_{i,j}}}_{w_1,\ \text{indep. on } i}(v_{i,j+1} - v_{i,j}) + \underbrace{\frac{r_{i,j-1} + r_{i,j}}{2\, m_{i,j}}}_{w_3,\ \text{indep. on } i}(v_{i,j-1} - v_{i,j}) + \underbrace{\frac{r_{i-1,j} + 2r_{i,j} + r_{i+1,j}}{2\, m_{i,j}}}_{w_2+w_4,\ \text{indep. on } i}(v_{i-1,j} - 2v_{i,j} + v_{i+1,j}).$$



Stencil (Δv and w) for the acceleration

$\leftarrow$ of an interior point (i, j)

of a boundary ($z-$axis) point $(0, j)$ $\rightarrow$



- **Now let the z-axis be the $i = 0$ ray. We apply reflection $v_{-1,j} = v_{1,j}$ and define $\frac{d\delta v_{0,j}}{dt}$ as**

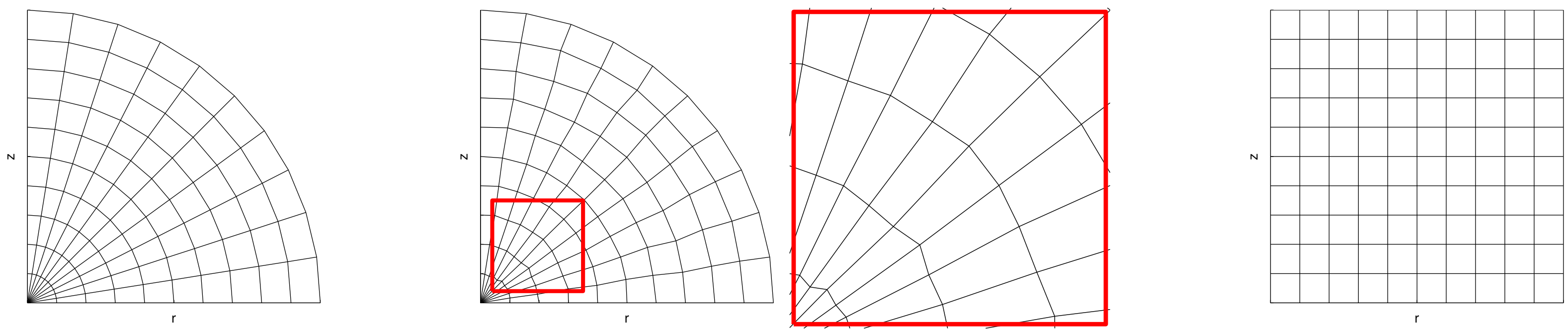
$$\underbrace{\frac{r_{1,j+1} + r_{1,j}}{2\, m_{1,j}}}_{w_1}(v_{0,j+1} - v_{0,j}) + \underbrace{\frac{r_{1,j-1} + r_{1,j}}{2\, m_{1,j}}}_{w_3}(v_{0,j-1} - v_{0,j}) + \underbrace{\frac{0 + 2r_{1,j} + r_{2,j}}{2\, m_{1,j}}}_{w_2+w_4} 2(v_{1,j} - v_{0,j}),$$

  where we used the "weights" w from the first off-axis node $(1, j)$, because these are not defined at $i = 0$ (on the axis). This is legal because w_1, w_3 and $w_2 + w_4$ don't depend on i
- **Such choice of acceleration of nodes on z-axis also makes sense for a general grid.**

Numerical Results - Overview

- **Methods:** LapEdge **(= ours) and for comparison** CS **(= Campbell-Shashkov incl. limiter as in the original paper [Campbell and Shashkov, JCP 172 (2001)])**
- **Test problems shown here: Noh and Sedov**
- **3 kinds of meshes:**

Equiangular polar | Perturbed polar ($\alpha = 0.05$) | Rectangular

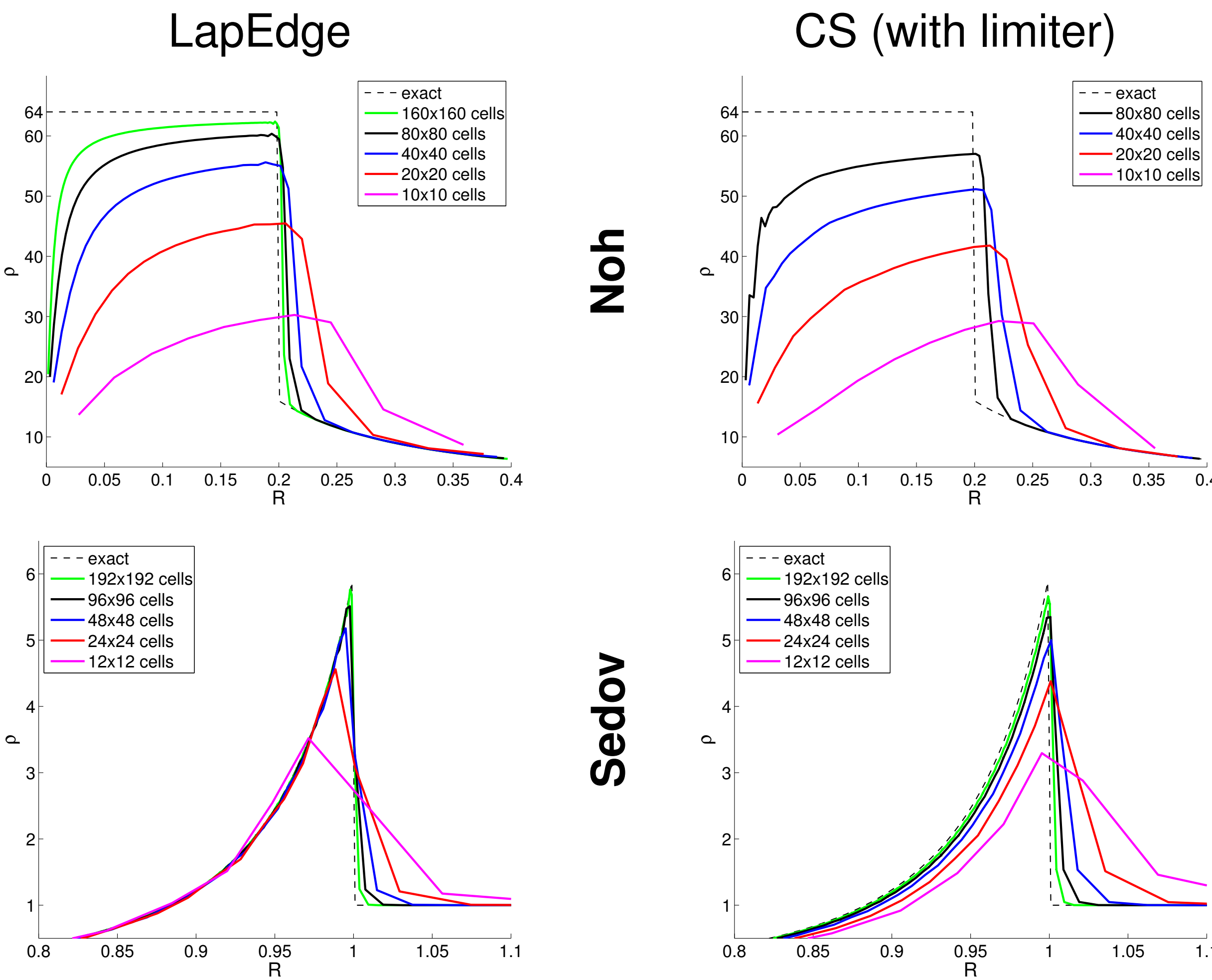


z-axis is the ray ($i = 0$), shown vertical at left in all figures

$\Rightarrow$ **on rect mesh:** **edges initially horizontal are formally circles with correction of $(\mathrm{f})_r$,**
edges initially vertical are formally rays without correction of $(\mathrm{f})_r$

- **Parameters (Kuropatenko, CFL, etc.):**
  - LapEdge**: All tests, meshes and resolutions run with exactly the same method** (no tuning)
  - CS**: Best results shown** (a lot of tuning)**, dissipation enforced by** *a posteriori* **cutoff of work**

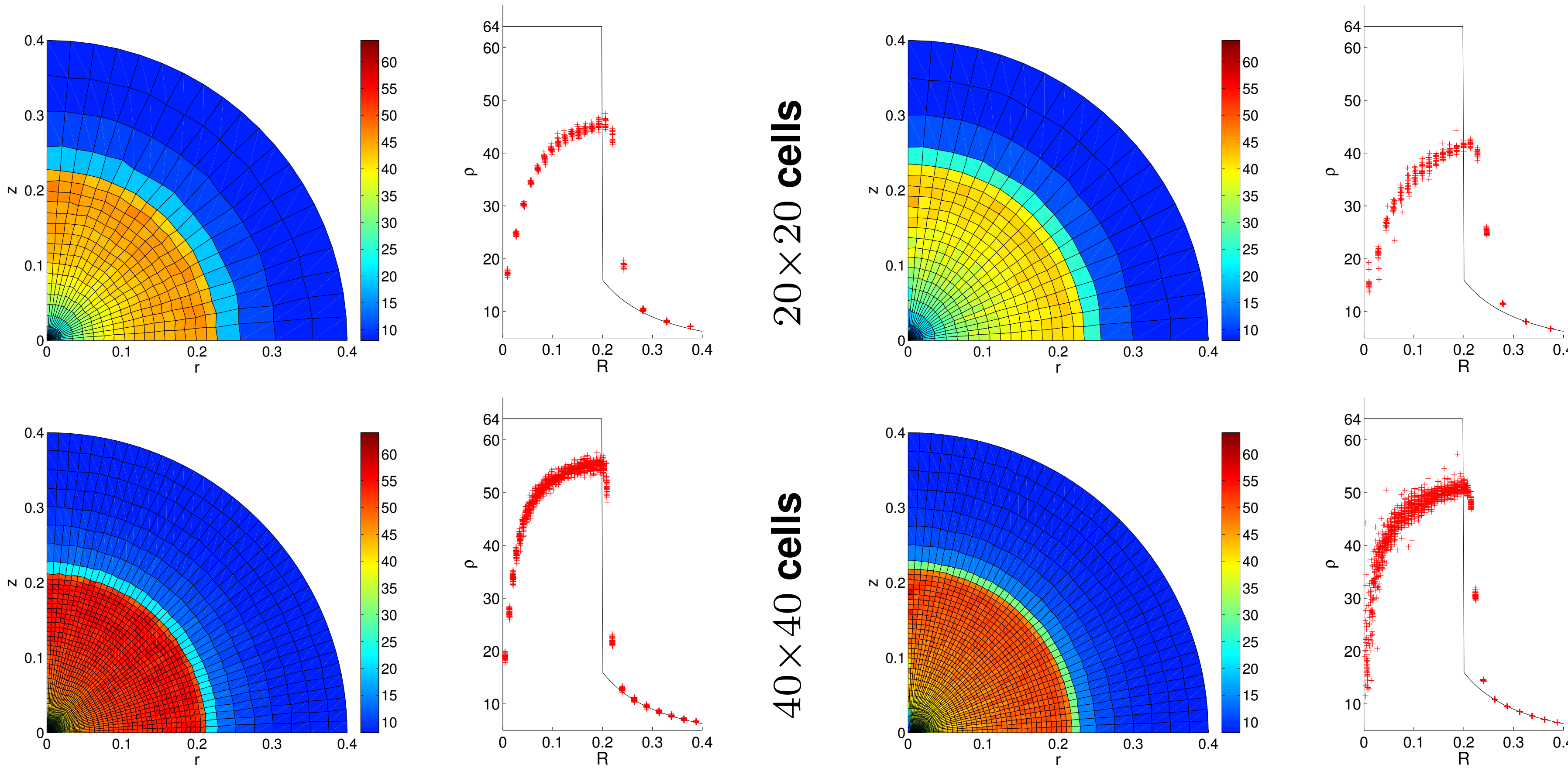
Numerical Results - Noh and Sedov, Symmetric Polar Grid



Numerical Results - Noh, Perturbed Polar Grid ($\alpha = 0.05$)

LapEdge

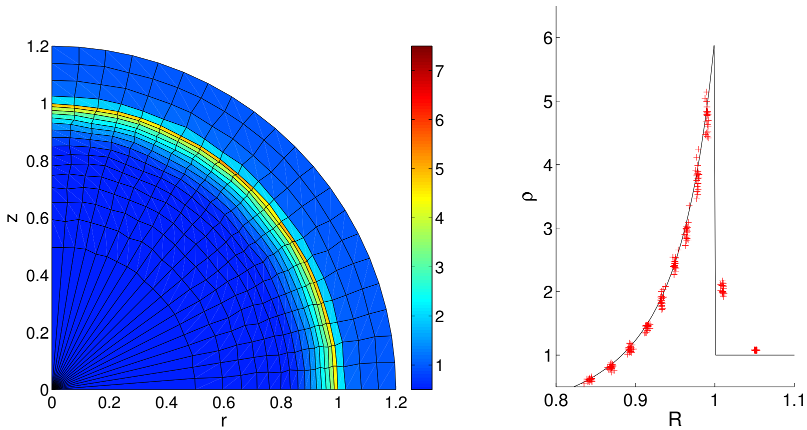
CS (with limiter)

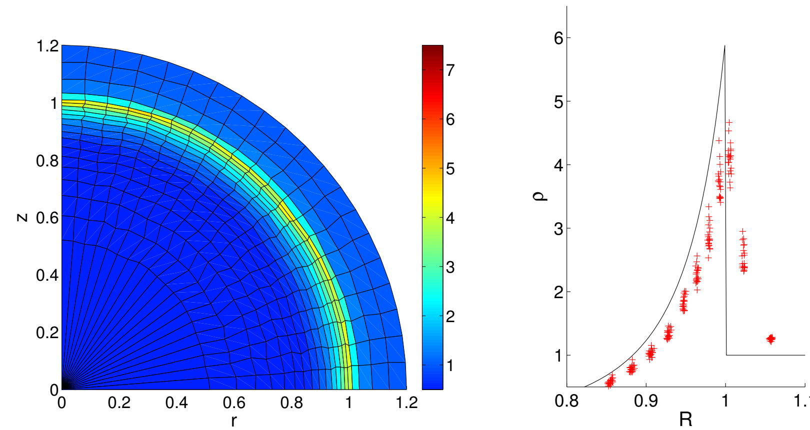


Numerical Results - Sedov, Perturbed Polar Grid ($\alpha = 0.075$)

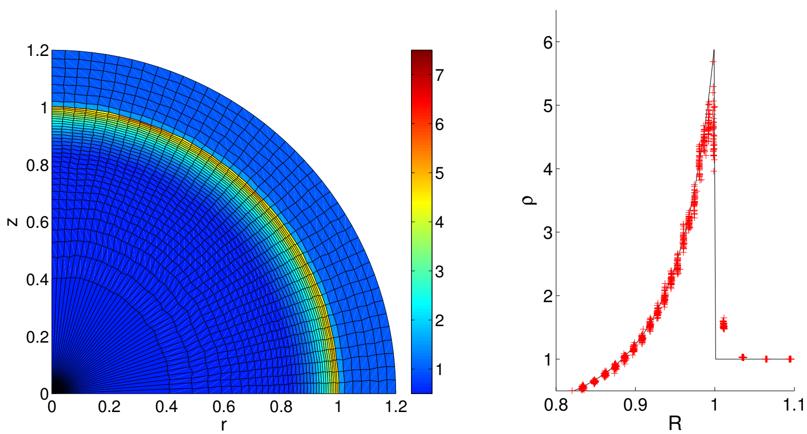
LapEdge

CS (with limiter)





20×20 **cells**

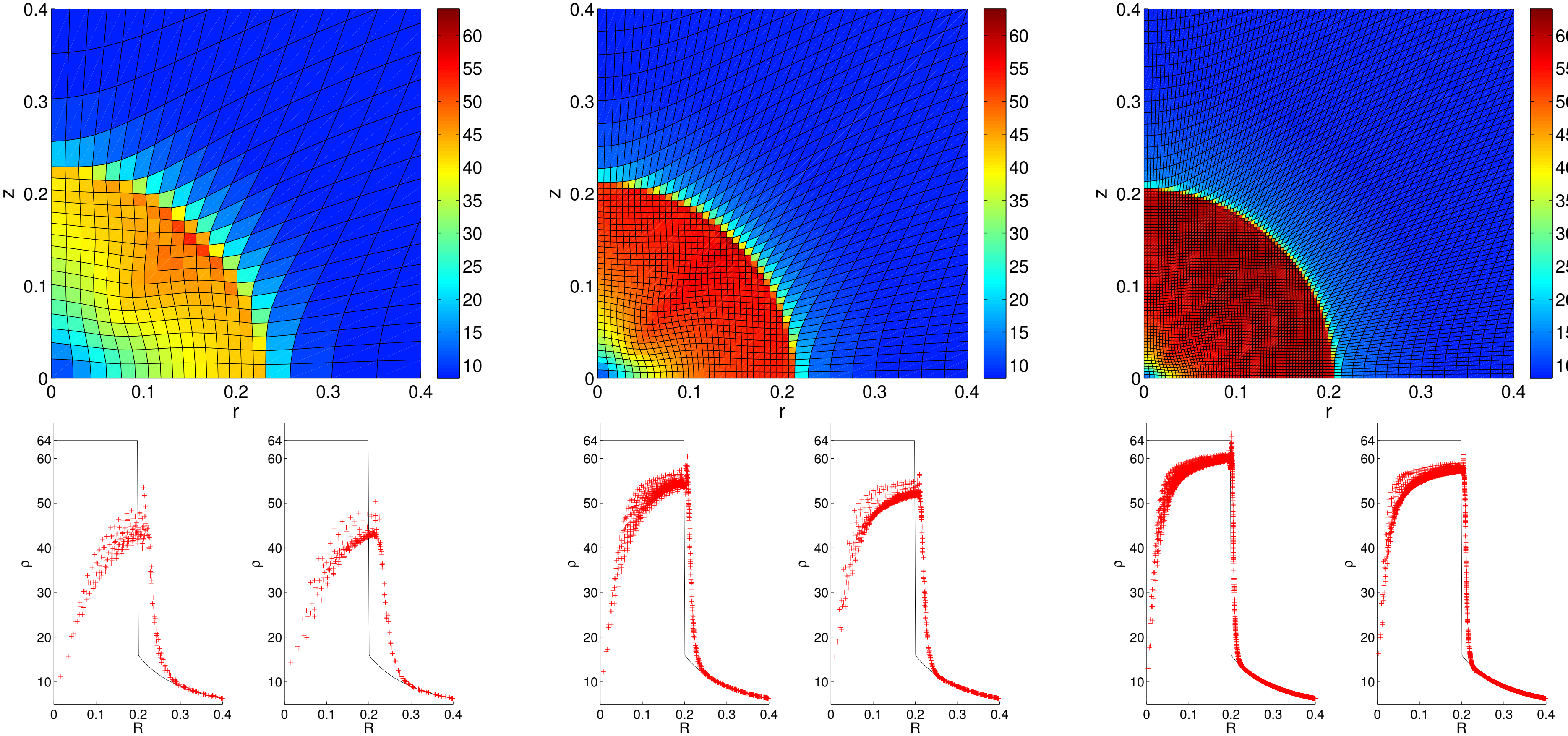


40×40 **cells**

FAILED

Numerical Results - Noh, Rectangular Grid

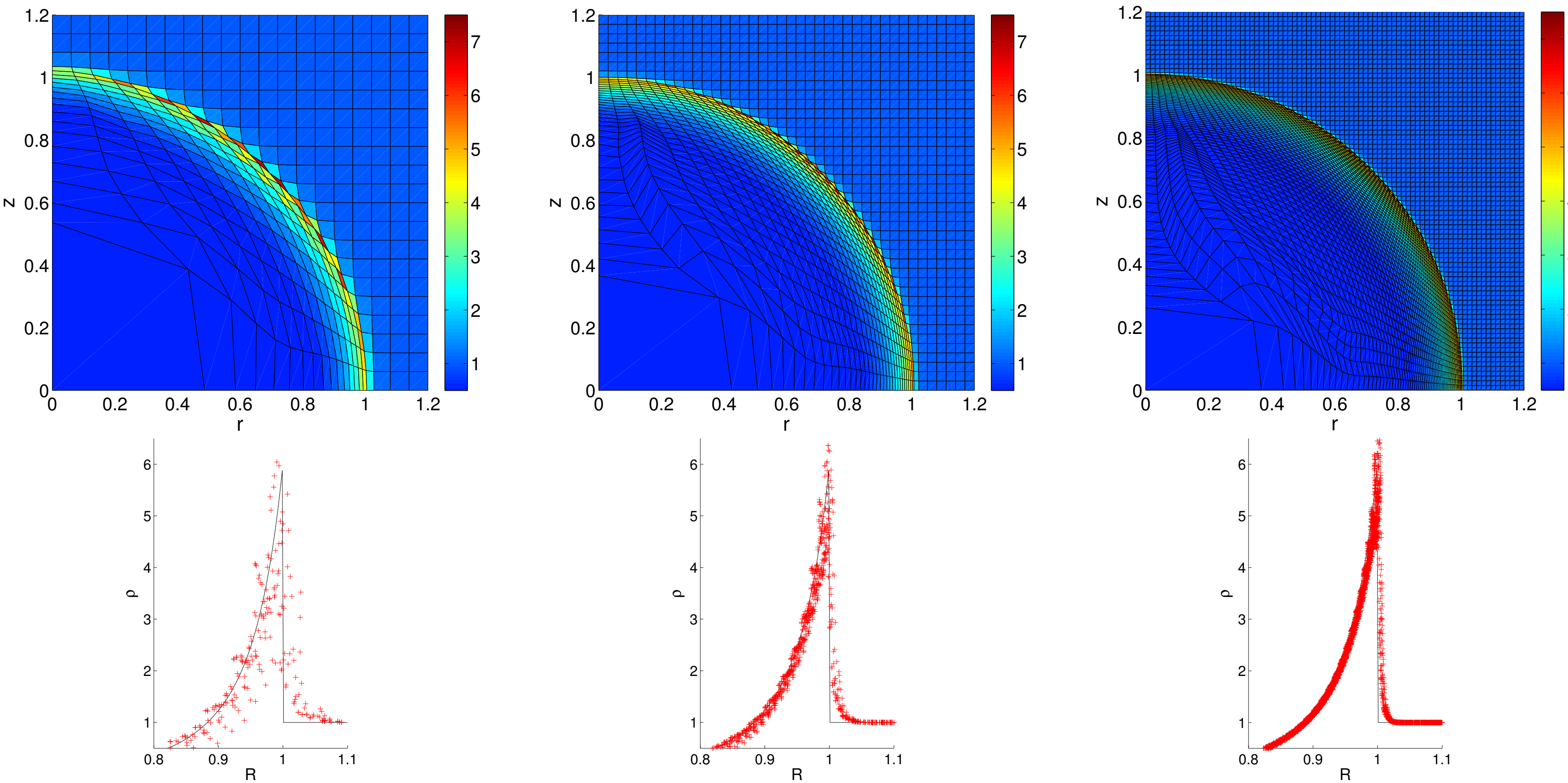
20×20 **cells** $\quad$ 40×40 **cells** $\quad$ 80×80 **cells**



Each column (= each resolution): Top and bottom left LapEdge, *bottom right* CS (with limiter)

Numerical Results - Sedov, Rectangular Grid, LapEdge

20×20 **cells** 40×40 **cells** 80×80 **cells**



Numerical Results - Other tests (done or planned)

- **Sanity checks: all OK**
  - **spherical problems shifted along z-axis** (polar meshes)
  - **"1D" Riemann problems along z-axis** (rectangular meshes)

- **Saltzman: results ugly**
  - **similar to** "CS90" $\Rightarrow$ **try Rayleigh-Taylor instability** to see if it kills vorticity
  - **using other subcells** (edge triangles, corner triangles) **helped a bit, but beyond scope**

- **Guderley (Lazarus)**
  - **non-shifted** (= mesh origin at the shock's center of convergence)**: OK (not interesting here)**
  - **shifted** ("off-axis")**: not tried yet** (planned for a related project)

Summary

- **Genuinely r-z** (not AW) **viscous force**
- **Always dissipative** (unlike AW) **and symmetry-preserving**
- **Applicable as is on various mesh topologies**
  (detects "circles" if they exist, provides acceptable visco if not)
- **No parameter tuning needed**
  (all tests and all resolutions with same CFL, same $k_1 = k_2 = 1, \ldots$)

Future Plans

- **Further tests, especially nonsymmetric test problems**
- **Simpler approximation of correction term**
- **Genuinely r-z pressure force with similar correction of the r-component**
  - symmetric and with very small violation of GCL
  - cell pressure forces done, now working on subcell forces

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- Comments and discussions:
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  - **Pierre-Henri Maire**

More Info

[P. Váchal, B. Wendroff: *A Symmetry Preserving Dissipative Artificial Viscosity in an r-z Staggered Lagrangian Discretization.* **Submitted to J. Comput. Phys.]**